

Complete Six-Gluon Disk Amplitude in Superstring Theory

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Abstract

We evaluate all next-to-maximal helicity violating (NMHV) six-gluon amplitudes in type I open superstring theory in four dimensions, at the disk level, to all orders in α' . Although the computation utilizes supersymmetric Ward identities, the result holds for all compactifications, even for those that break supersymmetry and is completely model-independent. Together with the maximally helicity violating (MHV) amplitudes presented in the previous work, our results provide the complete six-gluon disk amplitude.

1. Introduction

Multi-gluon scattering amplitudes are important from both theoretical and experimental points of view because they describe the processes underlying hadronic jet production at high energy colliders. Up to the energies accessible to the existing accelerators, there is an excellent agreement between experimental data and the amplitudes calculated in the framework of perturbative Quantum Chromodynamics (QCD). If in the upcoming Large Hadron Collider (LHC) experiments any discrepancy is discovered between QCD and the observed jet cross sections, it will be interpreted as a signal of new physics beyond the standard model.

Among the extensions of the standard model, superstring theory stands out as one of the boldest ones because it incorporates gravity and covers a huge span of energies, up to the Planck mass. However, the fundamental energy scale is the string mass, which need not necessarily to be as high provided that the Universe contains some large extra dimensions [1,2]. If the string scale is within, or not too far above the range of LHC energies, the effects of Regge excitations may be observable, and a direct experimental proof of superstring theory can be at hand. In particular, the multi-gluon amplitudes will be affected by the so-called α' corrections and the measurement of the corresponding jet cross sections can reveal some spectacular signals of superstring theory.

In a recent series of papers [3,4,5,6], we developed a formalism for computing N -gluon amplitudes at the disk level, *i.e.* at the leading order in the string/gauge coupling constant but to all orders in α' . The most important property of these leading contributions is that they are completely model-independent. These amplitudes are very robust because they hold for arbitrary compactifications of superstring theory from ten to four dimensions, including those that break supersymmetry.¹ The formalism combines the use of traditional supersymmetry and helicity techniques together with some elements of the theory of multiple hypergeometric integrals that allow expressing the final results in terms of $(N - 3)!$ generalized hypergeometric functions of kinematic invariants. This is particularly effective when applied to the maximally helicity violating [9] (MHV) amplitudes [6]. Next-to-maximal helicity violating (NMHV) amplitudes which appear starting from $N = 6$ seem to have a more complicated structure. In this work, we study the case of six-gluon NMHV amplitudes. We obtain some relatively simple expressions for NMHV amplitudes which complement the MHV amplitudes presented in [4,5,6], providing the full six-gluon disk amplitude.

¹ The difference between supersymmetric and non-supersymmetric amplitudes appears at the one-loop level. In particular, the presence of non-supersymmetric dimension six operator $\text{Tr}(F_{\mu\nu}F^{\nu\rho}F_{\rho\mu})$, where F is the gauge field strength tensor, may have some interesting phenomenological consequences [7,8].

The paper is organized as follows. In Section 2, we present supersymmetry (SUSY) relations that allow expressing all three independent six-gluon NMHV amplitudes in terms of the amplitudes involving four scalars and two gluons, four scalars and two gauginos, and six scalars. In Section 3, we evaluate these auxiliary amplitudes and express them in terms of certain hypergeometric integrals. In Section 4, we combine them according to SUSY relations and obtain explicit expressions for all NMHV amplitudes. We show that the leading order of the expansion in powers of α' correctly reproduces the QCD result. In two Appendices, we list the relevant hypergeometric functions and give their α' -expansions up to the next-to-leading order $\mathcal{O}(\alpha'^2)$ with respect to the leading (QCD) contributions.

2. SUSY Relations for NMHV Amplitudes

The full six-gluon NMHV amplitude can be constructed from three *partial* subamplitudes [10,11], each associated to the same Chan-Paton factor $\text{Tr}(T^{a_1} \cdots T^{a_6})$, but characterized by three inequivalent helicity orderings:

$$\begin{aligned} A^Y &\equiv A(g_1^-, g_2^-, g_3^+, g_4^+, g_5^-, g_6^+), \\ A^X &\equiv A(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^+), \\ A^Z &\equiv A(g_1^-, g_2^+, g_3^-, g_4^+, g_5^-, g_6^+). \end{aligned} \tag{2.1}$$

These amplitudes will be expressed in terms of the following amplitudes with the gluons replaced by scalars or fermions:

$$\begin{aligned} A_\lambda^Y &\equiv A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, \lambda_5^-, \lambda_6^+), & A_s^Y &\equiv A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, \phi_5^-, \phi_6^+), \\ A_\lambda^X &\equiv A(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-, \lambda_5^-, \lambda_6^+), & A_s^X &\equiv A(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-, \phi_5^-, \phi_6^+), \\ A_\lambda^Z &\equiv A(\phi_1^-, \phi_2^+, \phi_3^-, \phi_4^+, \lambda_5^-, \lambda_6^+), & A_s^Z &\equiv A(\phi_1^-, \phi_2^+, \phi_3^-, \phi_4^+, \phi_5^-, \phi_6^+), \end{aligned} \tag{2.2}$$

and the amplitudes with four gluons replaced by scalars:

$$\begin{aligned} A_g^Y &\equiv A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, g_5^-, g_6^+), \\ A_g^X &\equiv A(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-, g_5^-, g_6^+), \\ A_g^Z &\equiv A(\phi_1^-, \phi_2^+, \phi_3^-, \phi_4^+, g_5^-, g_6^+). \end{aligned} \tag{2.3}$$

Here, ϕ is the scalar component of $\mathcal{N} = 2$ gauge supermultiplet and λ is one of the two gauginos [6].

Recently, we showed [6] that all field-theoretical SUSY relations [12,13] between scattering amplitudes hold also in superstring theory at the disk level, to all orders in α' . Actually, we find most useful the relations already used in the original computation of the

six-gluon QCD amplitudes [14]. In order to write these relations down, we first introduce the following kinematic variables

$$\begin{aligned} Y &= k_3 + k_4 + k_6, & \alpha_Y &= -\langle 12 \rangle [34] [6|Y|5], & y &= \langle 12 \rangle [34] Y^2, \\ X &= k_1 + k_2 + k_6, & \alpha_X &= -[12] \langle 34 \rangle [6|X|5], & x &= [12] \langle 34 \rangle X^2, \\ Z &= k_2 + k_4 + k_6, & \alpha_Z &= -\langle 13 \rangle [24] [6|Z|5], & z &= \langle 13 \rangle [24] Z^2, \end{aligned} \quad (2.4)$$

depending on the momenta k_1, k_2, \dots, k_6 . Here, we used the standard notation [10,11] for spinor products, in particular:

$$[6|Y|5] = [63] \langle 35 \rangle + [64] \langle 45 \rangle, \quad \text{etc.} \quad (2.5)$$

For the scalar products of momenta, we use the notation of [5]:

$$s_{ij} = 2\alpha' k_i k_j, \quad s_i = \alpha' (k_i + k_{i+1})^2, \quad t_i = \alpha' (k_i + k_{i+1} + k_{i+2})^2 \quad (i + 6 \equiv i). \quad (2.6)$$

All scalar products s_{ij} can be expressed in terms of $s_1, s_2, s_3, s_4, s_5, s_6$ and t_1, t_2, t_3 [5], *e.g.* $s_{12} = s_1$ *etc.* Now the SUSY relations can be written as

$$\begin{aligned} A^Y &= \frac{\alpha'^4}{s_{12}^2 s_{34}^2} (y^2 A_g^Y - 2y \alpha_Y A_\lambda^Y + \alpha_Y^2 A_s^Y), \\ A^X &= \frac{\alpha'^4}{s_{12}^2 s_{34}^2} (x^2 A_g^X - 2x \alpha_X A_\lambda^X + \alpha_X^2 A_s^X), \\ A^Z &= \frac{\alpha'^4}{s_{13}^2 s_{24}^2} (z^2 A_g^Z - 2z \alpha_Z A_\lambda^Z + \alpha_Z^2 A_s^Z), \end{aligned} \quad (2.7)$$

where the factors α'^4 appear artificially, due to the choice of string units in (2.6). Note that the latter two relations (X, Z) follow from the first one (Y) by the replacements $(1 \leftrightarrow 4, 2 \leftrightarrow 3)$ and $(2 \leftrightarrow 3)$, respectively. In the next Section, we evaluate the amplitudes appearing on the r.h.s. of Eq. (2.7).

3. Six-point disk scattering of scalars, gauginos and vectors

In this Section, we compute the six-point string amplitudes (2.2) and (2.3) involving scalars, gauginos and vectors of the four-dimensional $\mathcal{N} = 2$ vector multiplet. The world-sheet of the string S-matrix is described by a disk with all external states Φ^a created by vertex operators V_{Φ^a} at the boundary the disk. In the notation of Refs. [4,5,6], the partial amplitude associated to the $\text{Tr}(T^{a_1} \dots T^{a_6})$ Chan-Paton factor takes the form:

$$\begin{aligned} A(\Phi^{a_1}, \Phi^{a_2}, \Phi^{a_3}, \Phi^{a_4}, \Phi^{a_5}, \Phi^{a_6}) &= V_{CKG}^{-1} \int_{z_1 < \dots < z_6} \left(\prod_{k=1}^6 dz_k \right) \\ &\times \langle V_{\Phi^{a_1}}(z_1) V_{\Phi^{a_2}}(z_2) V_{\Phi^{a_3}}(z_3) V_{\Phi^{a_4}}(z_4) V_{\Phi^{a_5}}(z_5) V_{\Phi^{a_6}}(z_6) \rangle. \end{aligned} \quad (3.1)$$

In order to cancel the total background ghost charge in the disk correlator (3.1), the vertex operators have to be chosen in the appropriate ghost picture. Furthermore, in Eq. (3.1), the factor V_{CKG} accounts for the volume of the conformal Killing group of the disk after choosing the conformal gauge. It will be canceled by fixing three vertex positions and introducing the respective c -ghost correlator. The gaugino vertex operators, in the $(-1/2)$ -ghost picture, are:²

$$\begin{aligned} V_{\lambda^a, I}^{(-1/2)}(z, u, k) &= g_\lambda T^a e^{-\phi/2} u^\alpha S_\alpha \Sigma^I e^{ik_\rho X^\rho} , \\ V_{\bar{\lambda}^a, I}^{(-1/2)}(z, u, k) &= g_\lambda T^a e^{-\phi/2} \bar{u}_{\dot{\beta}} S^{\dot{\beta}} \bar{\Sigma}^I e^{ik_\rho X^\rho} , \quad I = 1, 2 . \end{aligned} \quad (3.2)$$

Here $S_\alpha, S_{\dot{\alpha}}$ are the spin fields with the indices α (or $\dot{\alpha}$) denoting negative (positive) chirality in four dimensions. Furthermore, ϕ is the scalar bosonizing the superghost system. In the above definitions, T^a are the Chan–Paton factors accounting for the gauge degrees of freedom of the two open string ends. The on-shell constraints $k^2 = 0$, $\not{k}u = 0$ are imposed.

For $\mathcal{N} = 2$ space-time SUSY the internal SCFT splits into two pieces. One piece is the $c = 3$ superconformal algebra, which corresponds to a torus compactification with the two complex internal fermions $\Psi^\mp = e^{\pm iH_3}$. The second piece represents a $c = 6$ superconformal algebra, which contains the $SU(2)$ currents $J_3 = i\partial H$, $J^{12} = e^{i\sqrt{2}H}$ and $J^{21} = e^{-i\sqrt{2}H}$ [16]. The (internal) Ramond fields Σ^I may be expressed by these bosonic fields H_3 and H :

$$\Sigma^1 = e^{\frac{i}{2}H_3} e^{\frac{i}{\sqrt{2}}H} , \quad \Sigma^2 = e^{\frac{i}{2}H_3} e^{-\frac{i}{\sqrt{2}}H} . \quad (3.3)$$

Finally, the vertex operators $V_{\phi^a, \pm}(z, k)$ for the scalars and for the vectors $V_{A^a}(z, \xi, k)$ can be found in Section 3 of [6].

The six-point correlator in the integrand of (3.1) is evaluated by performing all possible Wick contractions. All three and four-point fermionic correlators involving fermions and spin fields are given in [17, 18, 6], while an important five-point correlator will be computed below. Because of the $PSL(2, \mathbf{R})$ invariance on the disk, we can fix three positions of the vertex operators. A convenient choice respecting the ordering $z_1 < \dots < z_6$ is

$$z_1 = -z_\infty = -\infty , \quad z_2 = 0 , \quad z_3 = 1 , \quad (3.4)$$

which implies the ghost factor $\langle c(z_1)c(z_2)c(z_3) \rangle = -z_\infty^2$. The remaining three vertex positions take arbitrary values inside the integration domain $1 < z_4 < z_5 < z_6 < \infty$. The

² The open string vertex couplings are $g_\phi = (2\alpha')^{1/2} g_{YM}$, $g_\lambda = (2\alpha')^{1/2} \alpha'^{1/4} g_{YM}$, and $g_A = (2\alpha')^{1/2} g_{YM}$ for the scalar, gaugino and vector, respectively. The $D = 4$ gauge coupling g_{YM} can be expressed in terms of the ten-dimensional gauge coupling g_{10} and the dilaton field ϕ_{10} through the relation $g_{YM} = g_{10} e^{\phi_{10}/2}$ [15].

latter is parameterized by $z_4 = x^{-1}$, $z_5 = (xy)^{-1}$ and $z_6 = (xyz)^{-1}$, with $0 < x, y, z < 1$. Generally for this choice, the integrand of (3.1) contains the common factor [3,5]

$$\begin{aligned} \mathcal{I}(x, y, z) = & x^{s_2} y^{t_2} z^{s_6} (1-x)^{s_3} (1-y)^{s_4} (1-z)^{s_5} \\ & \times (1-xy)^{t_3-s_3-s_4} (1-yz)^{t_1-s_4-s_5} (1-xyz)^{s_1+s_4-t_1-t_3} . \end{aligned} \quad (3.5)$$

The resulting integrals represent generalized Euler integrals and integrate to multiple Gaussian hypergeometric functions [3,5].

In order to correctly normalize the amplitudes, some additional factors have to be taken into account. They stem from determinants and Jacobians of certain path integrals. On the disk, the net result of those contributions is an additional factor of $C_{D_2} = \frac{1}{2g_{YM}^2 \alpha'^2}$ which must be included in all disk correlators [15].

3.1. Four scalars and two gauginos

First, we compute the three subamplitudes (2.2) involving four scalars and two gauginos: A_λ^Y , A_λ^X and A_λ^Z . To that end, we evaluate the amplitude (3.1) with the following correlator:

$$\langle V_{\phi^{a_1}}^{(0)}(z_1, k_1) V_{\phi^{a_2}}^{(0)}(z_2, k_2) V_{\phi^{a_3}}^{(0)}(z_3, k_3) V_{\phi^{a_4}}^{(-1)}(z_4, k_4) V_{\lambda^{a_5, I}}^{(-1/2)}(z_5, u_5, k_5) V_{\bar{\lambda}^{a_6, J}}^{(-1/2)}(z_6, \bar{u}_6, k_6) \rangle , \quad (3.6)$$

for the helicity configurations Y , X and Z .

To compute (3.6), we need³ the five-point function

$$\begin{aligned} \langle \psi^{\lambda_1}(z_1) \psi^{\lambda_2}(z_2) \psi^{\lambda_3}(z_3) S_\alpha(z_4) S_{\dot{\beta}}(z_5) \rangle = & \frac{1}{\sqrt{2}} (z_{14} z_{15} z_{24} z_{25} z_{34} z_{35})^{-1/2} \\ & \times \left\{ \frac{z_{24} z_{35}}{z_{23}} \sigma_{\alpha\dot{\beta}}^{\lambda_1} \delta^{\lambda_2 \lambda_3} - \frac{z_{14} z_{35}}{z_{13}} \sigma_{\alpha\dot{\beta}}^{\lambda_2} \delta^{\lambda_1 \lambda_3} + \frac{z_{14} z_{25}}{z_{12}} \sigma_{\alpha\dot{\beta}}^{\lambda_3} \delta^{\lambda_1 \lambda_2} + \frac{1}{2} z_{45} (\sigma^{\lambda_1} \bar{\sigma}^{\lambda_2} \sigma^{\lambda_3})_{\alpha\dot{\beta}} \right\} , \end{aligned} \quad (3.7)$$

which may be derived by studying its singular behavior and by using equations written in [6]. The last term in (3.7) may also be rewritten thanks to the identity⁴: $i\epsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda} \sigma_\lambda =$

³ Throughout this article we adapt to the notation and spinor algebra of the book of Wess and Bagger. In particular, spinor indices are raised and lowered with the anti-symmetric tensors $\epsilon_{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$. Besides spinor products are defined to be $\chi\eta = \chi^\alpha \epsilon_{\alpha\beta} \eta^\beta$ ($\bar{\chi}\bar{\eta} = \bar{\chi}_{\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\eta}_{\dot{\beta}}$) for some spinors χ, η ($\bar{\chi}, \bar{\eta}$).

⁴ With this identity the correlator (3.7) becomes:

$$\begin{aligned} \langle \psi^{\lambda_1}(z_1) \psi^{\lambda_2}(z_2) \psi^{\lambda_3}(z_3) S_\alpha(z_4) S_{\dot{\beta}}(z_5) \rangle = & \frac{1}{2\sqrt{2}} (z_{14} z_{15} z_{24} z_{25} z_{34} z_{35})^{-1/2} \{ i \epsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda} (\sigma_\lambda)_{\alpha\dot{\beta}} z_{45} \\ & + \frac{\sigma_{\alpha\dot{\beta}}^{\lambda_1} \delta^{\lambda_2 \lambda_3}}{z_{23}} (z_{24} z_{35} + z_{25} z_{34}) - \frac{\sigma_{\alpha\dot{\beta}}^{\lambda_2} \delta^{\lambda_1 \lambda_3}}{z_{13}} (z_{14} z_{35} + z_{15} z_{34}) + \frac{\sigma_{\alpha\dot{\beta}}^{\lambda_3} \delta^{\lambda_1 \lambda_2}}{z_{12}} (z_{14} z_{25} + z_{15} z_{24}) \} . \end{aligned}$$

$\sigma^{\lambda_1} \bar{\sigma}^{\lambda_2} \sigma^{\lambda_3} + \delta^{\lambda_2 \lambda_3} \sigma^{\lambda_1} - \delta^{\lambda_1 \lambda_3} \sigma^{\lambda_2} + \delta^{\lambda_1 \lambda_2} \sigma^{\lambda_3}$. Furthermore, we need the two correlators of internal fields:

$$\begin{aligned} \langle \Psi(z_1) \bar{\Psi}(z_2) \Sigma^I(z_3) \bar{\Sigma}^J(z_4) \rangle &= \delta^{IJ} z_{12}^{-1} z_{34}^{-3/4} \left(\frac{z_{13} z_{24}}{z_{14} z_{23}} \right)^{1/2}, \\ \langle \Psi(z_1) \Psi(z_2) \bar{\Psi}(z_3) \bar{\Psi}(z_4) \Sigma^I(z_5) \bar{\Sigma}^J(z_6) \rangle &= \delta^{IJ} \frac{z_{12} z_{34}}{z_{13} z_{14} z_{23} z_{24}} \left(\frac{z_{15} z_{25} z_{36} z_{46}}{z_{16} z_{26} z_{35} z_{45}} \right)^{1/2} z_{56}^{-3/4}. \end{aligned} \quad (3.8)$$

All remaining correlators appearing in (3.6) are basic and can be found in [6].

After assembling everything in the correlators (3.6), each of the partial amplitudes Y , X and Z takes the form

$$A_\lambda = 4\alpha'^2 g_{YM}^4 \left([6|1|5] L_1 + [6|2|5] L_2 + [6|3|5] L_3 - \alpha' [6|3|2][2|1|5] L_4 \right), \quad (3.9)$$

with the set of four functions $L_i \in \{L_i^Y, L_i^X, L_i^Z\}$, $i = 1, 2, 3, 4$, specific to the three helicity configurations Y , X and Z , respectively. The integral representations of these functions are given in Appendix A. Actually, the kinematic factor in front of L_4 can be expressed in terms of those in front of $L_{1,2,3}$, and the result (3.9) can be simplified to

$$A_\lambda = 4\alpha'^2 g_{YM}^4 \left([6|1|5] H_1 + [6|2|5] H_2 + [6|3|5] H_3 \right), \quad (3.10)$$

where:

$$\begin{aligned} H_1 &= L_1 - \frac{L_4}{2s_5} (s_{25}s_{36} - s_{26}s_{35} + s_{23}s_{56}), \\ H_2 &= L_2 - \frac{L_4}{2s_5} (s_{16}s_{35} - s_{15}s_{36} - s_{13}s_{56}), \\ H_3 &= L_3 - \frac{L_4}{2s_5} (s_{15}s_{26} - s_{16}s_{25} + s_{12}s_{56}). \end{aligned} \quad (3.11)$$

3.2. Six scalars

Here, we compute the three six-scalar subamplitudes (2.2): A_s^Y , A_s^X and A_s^Z . To that end, we evaluate the amplitude (3.1) with the correlators

$$\langle V_{\phi^{a_1}}^{(0)}(z_1, k_1) V_{\phi^{a_2}}^{(0)}(z_2, k_2) V_{\phi^{a_3}}^{(0)}(z_3, k_3) V_{\phi^{a_4}}^{(0)}(z_4, k_4) V_{\phi^{a_5}}^{(-1)}(z_5, k_5) V_{\phi^{a_6}}^{(-1)}(z_6, k_6) \rangle, \quad (3.12)$$

for the helicity configurations Y , X and Z . After a straightforward calculation, we obtain

$$A_s = 4\alpha' g_{YM}^4 L_5, \quad (3.13)$$

with the functions $L_5 \in \{L_5^Y, L_5^X, L_5^Z\}$ specific to the three helicity configurations Y , X and Z , respectively. Again, we present the integrals L_5 for the three cases in Appendix A.

3.3. Four scalars and two vectors

Finally, we evaluate the three subamplitudes (2.3) involving four scalars and two gauge fields: A_g^Y , A_g^X and A_g^Z . To that end, we evaluate the amplitude (3.1) with the correlators

$$\langle V_{\phi^{a_1}}^{(0)}(z_1, k_1) V_{\phi^{a_2}}^{(0)}(z_2, k_2) V_{\phi^{a_3}}^{(0)}(z_3, k_3) V_{\phi^{a_4}}^{(0)}(z_4, k_4) V_{A^{a_5}}^{(-1)}(z_5, \xi_5, k_5) V_{A^{a_6}}^{(-1)}(z_6, \xi_6, k_6) \rangle, \quad (3.14)$$

for the helicity configurations Y , X and Z . The amplitude A_g^Y has already been computed in [6]. In fact, all three (partial) amplitudes have a similar form:

$$\begin{aligned} A_g = 8\alpha'^2 g_{YM}^4 [& (\xi_5 k_3)(\xi_6 k_2) K_1 + (\xi_5 k_2)(\xi_6 k_3) K_2 + (\xi_5 k_1)(\xi_6 k_2) K_3 \\ & + (\xi_5 k_1)(\xi_6 k_3) K_4 + (\xi_5 k_2)(\xi_6 k_1) K_5 + (\xi_5 k_3)(\xi_6 k_1) K_6 \\ & + (\xi_5 k_3)(\xi_6 k_4) K_7 + (\xi_5 k_4)(\xi_6 k_3) K_8 + (\xi_5 k_2)(\xi_6 k_4) K_9 \\ & + (\xi_5 k_4)(\xi_6 k_2) K_{10} + (\xi_5 k_1)(\xi_6 k_4) K_{11} + (\xi_5 k_4)(\xi_6 k_1) K_{12} + (\xi_5 \xi_6) K_{13}] , \end{aligned} \quad (3.15)$$

specified by thirteen functions K_i for each configuration Y , X and Z . Here, ξ_5 and ξ_6 are the gluon polarization vectors. Actually, one also finds $A_g^Y = A_g^X$, therefore $K_i^Y = K_i^X$. In order to write Eq. (3.15) more explicitly, we choose k_6 as the reference vector for the (negative) polarization vector of gluon g_5 and k_5 as the reference vector for the (positive) polarization vector of gluon g_6 . Then

$$\xi_5^- \xi_6^+ = 0 \quad , \quad (\xi_5^- k_i)(\xi_6^+ k_j) = -\alpha' \frac{[6i]\langle i5\rangle[6j]\langle j5\rangle}{2s_5} , \quad (3.16)$$

and the amplitude (3.15) can be rewritten as

$$\begin{aligned} A_g = -\frac{4\alpha'^3 g_{YM}^4}{s_5} (& [6|2|5\rangle [6|3|5\rangle G_1 + [6|1|5\rangle [6|2|5\rangle G_2 + [6|1|5\rangle [6|3|5\rangle G_3 \\ & + [6|3|5\rangle [6|4|5\rangle G_4 + [6|2|5\rangle [6|4|5\rangle G_5 + [6|1|5\rangle [6|4|5\rangle G_6) , \end{aligned} \quad (3.17)$$

where $G_1 = K_1 + K_2$, $G_2 = K_3 + K_5$, $G_3 = K_4 + K_6$, $G_4 = K_7 + K_8$, $G_5 = K_9 + K_{10}$ and $G_6 = K_{11} + K_{12}$. The integral representations of all these functions are given in Appendix A.

4. Six-gluon NMHV Amplitudes

After computing all auxiliary amplitudes, we are now in a position to write down the six-gluon amplitudes. The result is obtained by substituting Eqs.(3.10), (3.13) and (3.17) into the r.h.s. of SUSY relations (2.7). We could leave this result as it is, however there are at least two good reasons for trying to combine all contributions into a more compact form. First, although for each helicity configuration, the amplitude depends on

ten functions G , H and L_5 , we know that only $(N-3)! = 6$ of them are independent [4,5,3]. Indeed, by using techniques developed in [3], it is possible to find relations between these functions. In what follows, we will combine the ten integrals G , H and L_5 into a set of six functions N_i , for each helicity configuration. Second, the kinematic factors appearing in auxiliary amplitudes are related, therefore they can be combined to a form involving fewer kinematic factors. This is highly desirable for many reasons, especially for the comparison of the $\alpha' = 0$ limit with the well-known QCD amplitudes. The six functions N_i will appear naturally in this context. In QCD, six-gluon NMHV amplitudes were first calculated in [14], and later recast in an elegant form in Refs.[19,10].⁵

The basic relations that allow combining various contributions on the r.h.s. of Eq. (2.7) are:

$$Y^2[16] + [15][6|Y|5] = -[12][6|Y|2] , \quad (4.1)$$

and its variations obtained by applying various permutations of $\{1, 2, 3, 4, 5, 6\}$ and/or complex conjugation. Eq. (4.1) follows from Schouten's identity and momentum conservation. The goal is to rewrite our results in a form similar to QCD amplitudes collected in Eqs.(5.28) and Table 4 of Ref. [10]. Below, we list the amplitudes obtained by manipulating kinematic factors on the r.h.s. of Eq. (2.7), for the three helicity configurations separately.

4.1. Y -configuration and its $\alpha' = 0$ limit

The result will be expressed in terms of the following kinematic variables:

$$\alpha_Y = -\langle 12 \rangle [34] [6|Y|5] , \quad \beta_Y = \langle 12 \rangle [46] [3|Y|5] , \quad \gamma_Y = \langle 51 \rangle [34] [6|Y|2] , \quad (4.2)$$

where α_Y , already defined in Eq. (2.4), is listed for completeness. Then

$$\begin{aligned} A(g_1^-, g_2^-, g_3^+, g_4^+, g_5^-, g_6^+) &= A^Y = \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5} T^{a_6}) \frac{(\sqrt{2} g_{YM})^4 \alpha'^5}{s_5} \\ &\times \left(N_1^Y \frac{\alpha_Y^2}{s_1^2 s_3^2} + N_2^Y \frac{\beta_Y^2}{s_1^2} + N_3^Y \frac{\gamma_Y^2}{s_3^2} + N_4^Y \frac{\alpha_Y \beta_Y}{s_1^2 s_3} + N_5^Y \frac{\alpha_Y \gamma_Y}{s_1 s_3^2} + N_6^Y \frac{\beta_Y \gamma_Y}{s_1 s_3} \right) , \end{aligned} \quad (4.3)$$

⁵ For more recent work on NMHV amplitudes, see *e.g.* Refs. [20,21,22,23,24].

with the functions N^Y written below:

$$\begin{aligned}
N_1^Y &= -(s_1 + s_3 + s_6 - t_2 - t_3) (s_1 + s_3 + s_4 - t_1 - t_3) G_1^Y - 2 s_5 (s_5 + s_6 - t_2) H_1^Y \\
&\quad - (s_1 + s_3 + s_6 - t_2 - t_3) [(s_5 + s_6 - t_2) G_2^Y - (s_4 + s_5 - t_1) G_5^Y - 2 s_5 H_2^Y] \\
&\quad + (s_1 + s_3 + s_4 - t_1 - t_3) [(s_5 + s_6 - t_2) G_3^Y - (s_4 + s_5 - t_1) G_4^Y - 2 s_5 H_3^Y] \\
&\quad - (s_4 + s_5 - t_1) (s_5 + s_6 - t_2) G_6^Y - s_5 L_5^Y , \\
N_2^Y &= -G_4^Y , \\
N_3^Y &= -G_2^Y , \\
N_4^Y &= (s_1 + s_3 + s_6 - t_2 - t_3) (G_1^Y - G_5^Y) + (s_1 + s_3 + 2 s_4 + s_5 - 2 t_1 - t_3) G_4^Y \\
&\quad - (s_5 + s_6 - t_2) (G_3^Y - G_6^Y) + 2 s_5 H_3^Y , \\
N_5^Y &= (s_1 + s_3 + s_4 - t_1 - t_3) (G_1^Y - G_3^Y) + (s_1 + s_3 + s_5 + 2 s_6 - 2 t_2 - t_3) G_2^Y \\
&\quad - (s_4 + s_5 - t_1) (G_5^Y - G_6^Y) + 2 s_5 (H_1^Y - H_2^Y) , \\
N_6^Y &= -G_1^Y + G_3^Y + G_5^Y - G_6^Y .
\end{aligned} \tag{4.4}$$

Their low-energy expansions are

$$\begin{aligned}
N_1^Y &= \frac{s_1 s_3 s_5}{s_4 s_6 t_3} - \zeta(2) \left(s_1 s_3 - \frac{s_1 s_3 t_1}{s_4} - \frac{s_1 s_3 t_2}{s_6} + \frac{s_1 s_2 s_3 s_5}{s_4 s_6} + \frac{s_1^2 s_3 s_5}{s_4 t_3} + \frac{s_1 s_3^2 s_5}{s_6 t_3} \right) + \dots , \\
N_2^Y &= \frac{s_1}{s_2 s_4 t_1} - \zeta(2) \left(\frac{s_1 s_6}{s_2 s_4} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_5}{s_2 t_1} \right) + \dots , \\
N_3^Y &= \frac{s_3}{s_2 s_6 t_2} - \zeta(2) \left(\frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2} \right) + \dots , \\
N_4^Y &= \frac{s_1 t_2}{s_2 s_4 s_6} - \zeta(2) \left(\frac{s_1 (s_1 - s_3 - s_5 - t_3)}{s_4} - \frac{s_1 t_2}{s_2} + \frac{s_1 t_1 t_2}{s_2 s_4} + \frac{s_1 t_2^2}{s_2 s_6} + \frac{s_1 t_2 t_3}{s_4 s_6} \right) + \dots , \\
N_5^Y &= \frac{s_3 t_1}{s_2 s_4 s_6} + \zeta(2) \left(\frac{s_3 (s_1 - s_3 + s_5 + t_3)}{s_6} + \frac{s_3 t_1}{s_2} - \frac{s_3 t_1^2}{s_2 s_4} - \frac{s_3 t_1 t_2}{s_2 s_6} - \frac{s_3 t_1 t_3}{s_4 s_6} \right) + \dots , \\
N_6^Y &= \frac{t_3}{s_2 s_4 s_6} + \zeta(2) \left(\frac{s_1 + s_3 - s_5}{s_2} - \frac{t_1 t_3}{s_2 s_4} - \frac{t_2 t_3}{s_2 s_6} - \frac{t_3^2}{s_4 s_6} \right) + \dots ,
\end{aligned} \tag{4.5}$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(3)\alpha'^3)$ with respect to the leading term. The above expansions have been obtained by collecting the expansions of G and L functions listed in Appendix A.

In the $\alpha' = 0$ limit of the amplitude (4.3), only the leading terms of Eq.(4.5) survive. Then all α' factors cancel and Eq. (4.3) agrees with the QCD amplitude written in Eq.(5.28) and Table 4 of Ref.[10].⁶

⁶ In order to compare, a cyclic permutation $\{1, 2, 3, 4, 5, 6\} \rightarrow \{3, 4, 5, 6, 1, 2\}$ must be performed on the result of Ref. [10].

4.2. X -configuration and its $\alpha' = 0$ limit

The result will be expressed in terms of the following kinematic variables:

$$\alpha_X = -[12]\langle 34 \rangle [6|X|5] , \quad \beta_X = [12]\langle 45 \rangle [6|X|3] , \quad \gamma_X = [61]\langle 34 \rangle [2|X|5] , \quad (4.6)$$

where α_X , already defined in Eq. (2.4), is listed for completeness. Then

$$\begin{aligned} A(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^+) &= A^X = \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5} T^{a_6}) \frac{(\sqrt{2} g_{YM})^4 \alpha'^5}{s_5} \\ &\times \left(N_1^X \frac{\alpha_X^2}{s_1^2 s_3^2} + N_2^X \frac{\beta_X^2}{s_1^2} + N_3^X \frac{\gamma_X^2}{s_3^2} + N_4^X \frac{\alpha_X \beta_X}{s_1^2 s_3} + N_5^X \frac{\alpha_X \gamma_X}{s_1 s_3^2} + N_6^X \frac{\beta_X \gamma_X}{s_1 s_3} \right) , \end{aligned} \quad (4.7)$$

with the functions N^X written below:

$$\begin{aligned} N_1^X &= -(s_4 - t_3) (s_6 - t_3) G_1^X - (s_6 - t_3) (s_6 G_2^X - s_4 G_5^X - 2 s_5 H_2^X) \\ &\quad + (s_4 - t_3) (s_6 G_3^X - s_4 G_4^X - 2 s_5 H_3^X) - s_6 (s_4 G_6^X + 2 s_5 H_1^X) - s_5 L_5^X , \\ N_2^X &= -G_4^X , \\ N_3^X &= -G_2^X , \\ N_4^X &= -(s_6 - t_3) (G_1^X - G_5^X) + s_6 (G_3^X - G_6^X) - (2 s_4 - t_3) G_4^X - 2 s_5 H_3^X , \\ N_5^X &= -(s_4 - t_3) (G_1^X - G_3^X) - (2 s_6 - t_3) G_2^X + s_4 (G_5^X - G_6^X) - 2 s_5 (H_1^X - H_2^X) , \\ N_6^X &= -G_1^X + G_3^X + G_5^X - G_6^X . \end{aligned} \quad (4.8)$$

Their low-energy expansions are

$$\begin{aligned} N_1^X &= -\zeta(2) s_1 s_3 + \dots , \\ N_2^X &= \frac{s_1}{s_2 s_4 t_1} - \zeta(2) \left(\frac{s_1 s_6}{s_2 s_4} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_5}{s_2 t_1} \right) + \dots , \\ N_3^X &= \frac{s_3}{s_2 s_6 t_2} - \zeta(2) \left(\frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2} \right) + \dots , \\ N_4^X &= \zeta(2) \left(\frac{s_1 t_2}{s_2} + \frac{s_1 t_3}{s_4} \right) + \dots , \\ N_5^X &= \zeta(2) \left(\frac{s_3 t_1}{s_2} + \frac{s_3 t_3}{s_6} \right) + \dots , \\ N_6^X &= \frac{t_3}{s_2 s_4 s_6} + \zeta(2) \left(\frac{s_1 + s_3 - s_5}{s_2} - \frac{t_1 t_3}{s_2 s_4} - \frac{t_2 t_3}{s_2 s_6} - \frac{t_3^2}{s_4 s_6} \right) + \dots , \end{aligned} \quad (4.9)$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(3)\alpha'^3)$ with respect to the leading (QCD) contribution. The $\alpha' = 0$ limit of the amplitude (4.7) agrees with Ref.[10].⁷ Note, in particular, that all terms multiplying α_X disappear in this limit.

⁷ In this case, a cyclic permutation $\{1, 2, 3, 4, 5, 6\} \rightarrow \{6, 1, 2, 3, 4, 5\}$ must be performed on the result of Ref. [10].

4.3. Z -configuration and its $\alpha' = 0$ limit

The result will be expressed in terms of the following kinematic variables:

$$\alpha_Z = -\langle 13 \rangle [24] [6|Z|5], \quad \beta_Z = \langle 13 \rangle [46] [2|Z|5], \quad \gamma_Z = \langle 51 \rangle [24] [6|Z|3], \quad (4.10)$$

where α_Z , already defined in Eq. (2.4), is listed for completeness. Then

$$A(g_1^-, g_2^+, g_3^-, g_4^+, g_5^-, g_6^+) = A^Z = \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5} T^{a_6}) \frac{(\sqrt{2} g_{YM})^4 \alpha'^5}{s_5} \\ \times \left(N_1^Z \frac{\alpha_Z^2}{s_{13}^2 s_{24}^2} + N_2^Z \frac{\beta_Z^2}{s_{13}^2} + N_3^Z \frac{\gamma_Z^2}{s_{24}^2} + N_4^Z \frac{\alpha_Z \beta_Z}{s_{13}^2 s_{24}} + N_5^Z \frac{\alpha_Z \gamma_Z}{s_{13} s_{24}^2} + N_6^Z \frac{\beta_Z \gamma_Z}{s_{13} s_{24}} \right), \quad (4.11)$$

with the functions N^Z written below:

$$N_1^Z = -(s_1 + s_2 + s_3 + s_6 - t_2 - t_3) (s_1 + s_2 + s_3 + s_4 - t_1 - t_3) G_1^Z \\ - (s_1 + s_2 + s_3 + s_6 - t_2 - t_3) [(s_5 + s_6 - t_2) G_2^Z - (s_4 + s_5 - t_1) G_5^Z - 2 s_5 H_2^Z] \\ + (s_1 + s_2 + s_3 + s_4 - t_1 - t_3) [(s_5 + s_6 - t_2) G_3^Z - (s_4 + s_5 - t_1) G_4^Z - 2 s_5 H_3^Z] \\ - (s_4 + s_5 - t_1) (s_5 + s_6 - t_2) G_6^Z - 2 s_5 (s_5 + s_6 - t_2) H_1^Z - s_5 L_5^Z, \\ N_2^Z = -G_5^Z, \\ N_3^Z = -G_3^Z, \\ N_4^Z = -(s_1 + s_2 + s_3 + s_4 - t_1 - t_3) (G_1^Z - G_4^Z + G_5^Z) - (s_5 + s_6 - t_2) (G_2^Z - G_5^Z - G_6^Z) \\ + 2 (s_4 - s_6 - t_1 + t_2) G_5^Z + 2 s_5 H_2^Z, \\ N_5^Z = -(s_1 + s_2 + s_3 + s_6 - t_2 - t_3) (G_1^Z - G_2^Z + G_3^Z) - (s_4 + s_5 - t_1) (G_3^Z + G_4^Z - G_6^Z) \\ + 2 (s_5 + s_6 - t_2) G_3^Z + 2 s_5 (H_1^Z - H_3^Z) \\ N_6^Z = -G_1^Z + G_2^Z + G_4^Z - G_6^Z. \quad (4.12)$$

The low-energy expansions of these functions are much more complicated in the two previous cases:

$$N_1^Z = (s_1 + s_2 - t_1) (s_2 + s_3 - t_2) \left(\frac{s_2}{s_1 s_3 t_3} + \frac{s_5}{s_4 s_6 t_3} + \frac{t_1}{s_1 s_4 t_3} + \frac{t_2}{s_3 s_6 t_3} \right) + \dots, \\ N_2^Z = (s_1 + s_2 - t_1) \left(\frac{1}{s_1 s_3 t_3} + \frac{1}{s_1 s_4 t_3} + \frac{1}{s_1 s_4 t_1} + \frac{1}{s_2 s_4 t_1} \right) +, \\ N_3^Z = (s_2 + s_3 - t_2) \left(\frac{1}{s_1 s_3 t_3} + \frac{1}{s_3 s_6 t_3} + \frac{1}{s_2 s_6 t_2} + \frac{1}{s_3 s_6 t_2} \right) + \dots, \\ N_4^Z = (s_1 + s_2 - t_1) \left(\frac{2s_2}{s_1 s_3 t_3} + \frac{s_2}{s_1 s_4 t_3} + \frac{1}{s_1 s_4} - \frac{s_5}{s_1 s_4 t_3} - \frac{s_6}{s_1 s_4 t_3} + \frac{t_1}{s_1 s_4 t_3} - \frac{t_2}{s_1 s_3 t_3} \right. \\ \left. + \frac{t_2}{s_3 s_6 t_3} + \frac{t_2}{s_4 s_6 t_3} + \frac{1}{s_1 t_3} + \frac{s_3}{s_1 s_4 t_3} - \frac{1}{s_4 t_3} + \frac{t_2}{s_4 s_6 s_2} \right) + \dots, \quad (4.13)$$

$$\begin{aligned}
N_5^Z &= (s_2 + s_3 - t_2) \left(\frac{s_1}{s_3 s_6 t_3} + \frac{t_1}{s_2 s_4 s_6} + \frac{1}{s_3 s_6} + \frac{t_1}{s_4 s_6 t_3} + \frac{t_2}{s_3 s_6 t_3} + \frac{1}{s_3 t_3} - \frac{s_4}{s_3 s_6 t_3} \right. \\
&\quad \left. - \frac{s_5}{s_3 s_6 t_3} + \frac{s_2}{s_3 s_6 t_3} - \frac{1}{s_6 t_3} - \frac{t_1}{s_3 t_3 s_1} + \frac{t_1}{s_4 t_3 s_1} + \frac{2s_2}{s_3 t_3 s_1} \right) + \dots, \\
N_6^Z &= \frac{2s_2}{s_1 s_3 t_3} + \frac{s_2}{s_1 s_4 t_3} + \frac{s_2}{s_3 s_6 t_3} + \frac{s_2}{s_4 s_6 t_3} + \frac{1}{s_1 s_4} + \frac{1}{s_3 s_6} + \frac{2}{s_4 s_6} + \frac{s_5}{s_1 s_3 t_3} - \frac{s_6}{s_1 s_4 t_3} \\
&\quad - \frac{t_1}{s_1 s_3 t_3} - \frac{t_2}{s_1 s_3 t_3} - \frac{1}{s_4 t_3} - \frac{s_4}{s_3 s_6 t_3} - \frac{1}{s_6 t_3} + \frac{t_3}{s_4 s_6 s_2} - \frac{1}{s_4 s_2} - \frac{1}{s_6 s_2} + \dots,
\end{aligned}$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(2)\alpha'^2)$ with respect to the leading (QCD) contributions to the amplitude. The order $\zeta(2)\alpha'^2$ is presented in Appendix B.

For this helicity configuration, the comparison of the $\alpha' = 0$ limit with QCD is a highly nontrivial and tedious exercise in spinor algebra which, fortunately, has a happy end. Most likely, another SUSY relation would be more efficient in handling this case.

More details on the functions N_i are given in Appendix B.

5. Summary and Outlook

Together with the MHV amplitudes presented in Refs. [4,5,6], the NMHV amplitudes presented in this work provide the complete six-gluon disk amplitude. As expected, the NMHV case is considerably more complex than MHV. Six gluons are still manageable (as well as seven-gluon MHVs [6]), but clearly more efficient techniques need to be developed for handling larger numbers of external gluons. To that end, some type of recursion relations should be constructed, similar to Berends–Giele relations [25] in QCD and/or to the so called MHV or recursive rules [26,21,27]. This is quite an involved task: all string excitations propagate in intermediate channels of the disk diagram, therefore a part of the problem is to extend string propagation off mass-shell.

In addition to possible phenomenological applications of our results already stressed in the Introduction, we should point out that the complete six-gluon string amplitude, together with the previously obtained five- and four-gluon amplitudes (summarized in Refs.[4,5]), provide all information necessary for constructing the non-Abelian Born–Infeld action up to the order $\mathcal{O}(\alpha'^4 F^6)$ in the gauge field strength F . Furthermore, a direct comparison of type I disk amplitudes with two-loop heterotic amplitudes – a non-trivial test of type I-heterotic duality [28] – becomes now possible.

It is interesting that multi-gluon disk amplitudes exhibit transcendentality behavior in their low-energy α' -expansions. Each power α'^n comes with the factor $\zeta(n)$, a product of zeta functions or multiple zeta values having transcendentality degree n .

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Appendix A. Hypergeometric functions L_i, G_j and their α' expansions

In this Appendix, we collect the hypergeometric functions L_i and G_i describing the auxiliary amplitudes (3.9), (3.13) and (3.17).

A.1. Helicity configuration Y

The functions L_i entering (3.9) are:

$$\begin{aligned}
L_1^Y &= s_{23} L_4^Y - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(s_{23} + \frac{1-s_{23}}{x} \right) \frac{\mathcal{I}(x, y, z)}{xyz(1-y)(1-z)} , \\
L_2^Y &= -s_{13} L_4^Y - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - s_{13} + \frac{s_{13}}{x} \right) \frac{\mathcal{I}(x, y, z)}{(1-y)(1-z)} , \\
L_3^Y &= s_{12} L_4^Y + s_{12} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{1-x}{1-z} \frac{\mathcal{I}(x, y, z)}{x(1-y)(1-xy)} , \\
L_4^Y &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{(1-x) \mathcal{I}(x, y, z)}{xz(1-y)(1-xy)} .
\end{aligned} \tag{A.1}$$

Their low-energy expansions are:

$$\begin{aligned}
L_1^Y &= -\frac{1}{s_2 s_4} - \frac{1}{s_2 s_5} - \frac{1}{s_2 s_6} + \frac{1}{s_4 s_6} + \frac{s_1}{s_2 s_4 t_1} + \frac{s_1}{s_2 s_5 t_1} + \frac{s_3}{s_2 s_5 t_2} + \frac{s_3}{s_2 s_6 t_2} + \frac{t_3}{s_2 s_4 s_6} + \dots , \\
L_2^Y &= -\frac{1}{s_2 s_4} - \frac{1}{s_2 s_5} + \frac{1}{s_4 s_6} + \frac{s_1}{s_2 s_4 s_6} + \frac{s_1}{s_2 s_4 t_1} + \frac{s_1}{s_2 s_5 t_1} - \frac{t_1}{s_2 s_4 s_6} + \dots , \\
L_3^Y &= \frac{s_1}{s_2 s_4 s_6} + \frac{s_1}{s_2 s_4 t_1} + \frac{s_1}{s_2 s_5 t_1} + \dots , \quad L_4^Y = \frac{1}{s_2 s_4 s_6} + \dots .
\end{aligned} \tag{A.2}$$

The function L_5 entering (3.13) is:

$$\begin{aligned}
L_5^Y &= - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[\left(1 - s_{13} \frac{1-xyz}{z(1-xy)} \right) \left(1 - s_{24} \frac{1-yz}{1-y} \right) \right. \\
&\quad + \frac{1}{x^2} \left(1 - s_{14} \frac{1-yz}{z(1-y)} \right) \left(1 - s_{23} \frac{1-xyz}{1-xy} \right) \\
&\quad \left. + \frac{(1-yz)(1-xyz)}{xz(1-y)(1-xy)} (s_{12}s_{34} - s_{14}s_{23} - s_{13}s_{24}) \right] \frac{\mathcal{I}(x, y, z)}{y(1-z)^2} .
\end{aligned} \tag{A.3}$$

It has the following α' -expansion:

$$\begin{aligned}
L_5^Y = & \frac{1}{s_2 s_4} \left(s_1 - s_3 + s_5 - \frac{s_1 s_5}{t_1} - t_1 \right) + \frac{1}{s_2 s_6} \left(-s_1 + s_3 + s_5 - \frac{s_3 s_5}{t_2} - t_2 \right) \\
& + \frac{1}{s_4 s_6} \left(s_1 + s_3 - s_5 - \frac{s_1 s_3}{t_3} - t_3 \right) - \frac{1}{s_2 s_4 s_6} (s_3 t_1 + s_1 t_2 + s_5 t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3) \\
& - \frac{1}{s_2 s_5} \left(\frac{s_1 s_4}{t_1} + \frac{s_3 s_6}{t_2} - t_3 \right) + \frac{2}{s_2} \left(1 - \frac{s_1}{t_1} - \frac{s_3}{t_2} \right) + \dots .
\end{aligned} \tag{A.4}$$

The functions G_i entering (3.17) are:

$$\begin{aligned}
G_1^Y &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\frac{1 - s_{14}}{x} + s_{14} \right) \frac{xy \mathcal{I}(x, y, z)}{(1 - xy)(1 - xyz)} , \\
G_2^Y &= -s_{34} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{xyz} , \\
G_3^Y &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - s_{24} + \frac{s_{24}}{x} \right) \frac{\mathcal{I}(x, y, z)}{yz(1 - xy)(1 - xyz)} , \\
G_4^Y &= -s_{12} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{y(1 - x)^2 \mathcal{I}(x, y, z)}{x(1 - y)(1 - xy)(1 - yz)(1 - xyz)} , \\
G_5^Y &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - s_{13} + \frac{s_{13}}{x} \right) \frac{y \mathcal{I}(x, y, z)}{(1 - y)(1 - yz)} , \\
G_6^Y &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\frac{1 - s_{23}}{x} + s_{23} \right) \frac{\mathcal{I}(x, y, z)}{xyz(1 - y)(1 - yz)} .
\end{aligned} \tag{A.5}$$

Their low-energy expansions are

$$\begin{aligned}
G_1^Y &= \zeta(2) + \dots , \quad G_2^Y = -\frac{s_3}{s_2 s_6 t_2} + \zeta(2) \left(\frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2} \right) + \dots , \\
G_3^Y &= \frac{1}{s_2 s_6} - \frac{s_3}{s_2 s_6 t_2} + \zeta(2) \left(1 - \frac{s_5}{s_2} - \frac{s_3}{s_6} - \frac{t_3}{s_6} + \frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2} - \frac{s_4 t_2}{s_2 s_6} \right) + \dots , \\
G_4^Y &= -\frac{s_1}{s_2 s_4 t_1} + \zeta(2) \left(\frac{s_1 s_6}{s_2 s_4} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_5}{s_2 t_1} \right) + \dots , \\
G_5^Y &= \frac{1}{s_2 s_4} - \frac{s_1}{s_2 s_4 t_1} + \zeta(2) \left(1 - \frac{s_1}{s_4} - \frac{s_5}{s_2} - \frac{t_3}{s_4} + \frac{s_1 s_6}{s_2 s_4} + \frac{s_1 s_5}{s_2 t_1} + \frac{s_1^2}{s_4 t_1} - \frac{s_6 t_1}{s_2 s_4} \right) + \dots ,
\end{aligned} \tag{A.6}$$

$$G_6^Y = \frac{1}{s_2 s_4} + \frac{1}{s_2 s_6} - \frac{s_1}{s_2 s_4 t_1} - \frac{t_3}{s_2 s_4 s_6} - \frac{s_3}{s_2 s_6 t_2} + \zeta(2) \left(1 - \frac{s_1}{s_2} - \frac{s_3}{s_2} - \frac{s_5}{s_2} - \frac{s_1}{s_4} - \frac{t_3}{s_4} - \frac{s_3}{s_6} \right. \\ \left. - \frac{t_3}{s_6} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_6}{s_2 s_4} + \frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_1 s_5}{s_2 t_1} + \frac{t_3^2}{s_4 s_6} - \frac{s_6 t_1}{s_2 s_4} - \frac{s_4 t_2}{s_2 s_6} + \frac{t_1 t_3}{s_2 s_4} + \frac{t_2 t_3}{s_2 s_6} + \frac{s_3^2}{s_6 t_2} \right) + \dots,$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(3)\alpha'^3)$ with respect to the leading (QCD) contribution.

A.2. Helicity configuration X

The functions L_i entering (3.9) are:

$$L_1^X = - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(s_{23} + \frac{1 - s_{23}}{x} \right) \frac{\mathcal{I}(x, y, z)}{xy(1-z)(1-yz)}, \quad (\text{A.7})$$

$$L_2^X = - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - s_{13} + \frac{s_{13}}{x} \right) \frac{\mathcal{I}(x, y, z)}{(1-z)(1-yz)},$$

$$L_3^X = s_{12} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{1-x}{1-z} \frac{\mathcal{I}(x, y, z)}{x(1-yz)(1-xyz)},$$

$$L_4^X = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{(1-x) \mathcal{I}(x, y, z)}{x(1-yz)(1-xyz)}.$$

Their low-energy expansions are:

$$L_1^X = -\frac{1}{s_2 s_5} + \frac{s_1}{s_2 s_5 t_1} + \frac{s_3}{s_2 s_5 t_2} + \dots, \quad L_2^X = -\frac{1}{s_2 s_5} + \frac{s_1}{s_2 s_5 t_1} + \dots, \quad (\text{A.8})$$

$$L_3^X = \frac{s_1}{s_2 s_5 t_1} + \dots, \quad L_4^X = \dots.$$

The function L_5 entering (3.13) is

$$L_5^X = - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[\left(1 - s_{13} \frac{z(1-xy)}{(1-xyz)} \right) \left(1 - s_{24} \frac{1-y}{1-yz} \right) \right. \\ \left. + \frac{1}{x^2} \left(1 - s_{14} \frac{(1-y)z}{1-yz} \right) \left(1 - s_{23} \frac{1-xy}{1-xyz} \right) \right. \\ \left. + \frac{(1-y)(1-xy)z}{x(1-yz)(1-xyz)} (s_{12}s_{34} - s_{14}s_{23} - s_{13}s_{24}) \right] \frac{\mathcal{I}(x, y, z)}{y(1-z)^2}. \quad (\text{A.9})$$

It has the following α' -expansion:

$$L_5^X = \frac{t_3}{s_2 s_5} - \frac{s_1 s_4}{s_2 s_5 t_1} - \frac{s_3 s_6}{s_2 s_5 t_2} + \dots \quad (\text{A.10})$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(2)\alpha'^2)$ with respect to the leading (QCD) contribution.

As already mentioned before $A_g^X = A_g^Y$, therefore the functions $G_i^X = G_i^Y$, see Eq.(A.5).

A.3. Helicity configuration Z

The functions L_i entering (3.9) are:

$$\begin{aligned} L_1^Z &= - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - \frac{s_{23}}{1-x} \right) \frac{\mathcal{I}(x, y, z)}{x^2 y z (1-y)(1-z)} , \\ L_2^Z &= -s_{13} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{x z (1-x)(1-y)(1-z)} , \\ L_3^Z &= - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - \frac{s_{12}}{x z} \right) \frac{\mathcal{I}(x, y, z)}{(1-x)(1-y)(1-z)(1-xyz)} , \\ L_4^Z &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{x z (1-x)(1-y)(1-xyz)} . \end{aligned} \quad (\text{A.11})$$

Their low-energy expansions are:

$$\begin{aligned} L_1^Z &= \frac{s_1}{s_2 s_4 t_1} + \frac{s_1}{s_2 s_5 t_1} + \frac{t_3}{s_2 s_4 s_6} + \frac{s_3}{s_2 s_5 t_2} + \frac{s_2}{s_3 s_5 t_2} + \frac{s_3}{s_2 s_6 t_2} + \frac{s_2}{s_3 s_6 t_2} + \frac{s_2}{s_3 s_6 t_3} \\ &\quad + \frac{s_2}{s_4 s_6 t_3} + \frac{s_2}{s_1 s_3 s_5} + \frac{s_2}{s_1 s_4 t_1} + \frac{s_2}{s_1 s_5 t_1} + \frac{s_2}{s_1 s_3 t_3} + \frac{s_2}{s_1 s_4 t_3} - \frac{1}{s_2 s_4} - \frac{1}{s_2 s_5} - \frac{1}{s_2 s_6} \\ &\quad + \frac{2}{s_4 s_6} + \frac{2}{s_4 t_1} + \frac{2}{s_5 t_1} + \frac{2}{s_5 t_2} + \frac{2}{s_6 t_2} + \dots , \\ L_2^Z &= (s_1 + s_2 - t_1) \left(\frac{1}{s_2 s_4 s_6} + \frac{1}{s_2 s_4 t_1} + \frac{1}{s_1 s_5 t_1} + \frac{1}{s_2 s_5 t_1} + \frac{1}{s_3 s_6 t_3} + \frac{1}{s_4 s_6 t_3} + \frac{1}{s_1 s_3 s_5} \right. \\ &\quad \left. + \frac{1}{s_1 s_3 t_3} + \frac{1}{s_1 s_4 t_1} + \frac{1}{s_1 s_4 t_3} \right) + \dots , \\ L_3^Z &= \frac{s_1}{s_2 s_4 s_6} + \frac{s_1}{s_2 s_4 t_1} + \frac{s_1}{s_2 s_5 t_1} + \frac{s_1}{s_3 s_6 t_3} + \frac{s_1}{s_4 s_6 t_3} + \frac{t_2}{s_1 s_3 s_5} + \frac{s_2}{s_1 s_4 t_1} + \frac{s_2}{s_1 s_5 t_1} + \frac{s_6}{s_1 s_3 t_3} \\ &\quad + \frac{s_6}{s_1 s_4 t_3} - \frac{1}{s_1 s_5} + \frac{1}{s_3 s_5} - \frac{1}{s_1 s_3} - \frac{1}{s_1 s_4} + \frac{2}{s_4 t_1} + \frac{2}{s_5 t_1} + \frac{2}{s_3 t_3} + \frac{2}{s_4 t_3} + \dots , \\ L_4^Z &= \frac{1}{s_2 s_4 s_6} + \frac{1}{s_1 s_3 t_3} + \frac{1}{s_1 s_4 t_3} + \frac{1}{s_3 s_6 t_3} + \frac{1}{s_4 s_6 t_3} + \dots . \end{aligned} \quad (\text{A.12})$$

The function L_5 entering (3.13) is

$$L_5^Z = - \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[\frac{1}{(1-x)^2} \left(1 - \frac{s_{12}}{z} \right) \left(1 - s_{34} \frac{(1-xy)(1-yz)}{(1-y)(1-xyz)} \right) \right. \\ \left. + \frac{1}{x^2} \left(1 - s_{14} \frac{1-yz}{(1-y)z} \right) \left(1 - s_{23} \frac{1-xy}{1-xyz} \right) \right. \\ \left. + \frac{(1-xy)(1-yz)}{xz(1-x)(1-y)(1-xyz)} (s_{12}s_{34} + s_{14}s_{23} - s_{13}s_{24}) \right] \frac{\mathcal{I}(x, y, z)}{y(1-z)^2}. \quad (\text{A.13})$$

It has the following expansion:

$$L_5^Z = \frac{1}{s_2 s_4} \left(s_1 - s_3 + s_5 - \frac{s_1 s_5}{t_1} - t_1 \right) + \frac{1}{s_2 s_6} \left(-s_1 + s_3 + s_5 - \frac{s_3 s_5}{t_2} - t_2 \right) \\ + \frac{1}{s_4 s_6} \left(s_1 + s_3 - s_5 - \frac{s_1 s_3}{t_3} - t_3 \right) - \frac{1}{s_2 s_4 s_6} (s_3 t_1 + s_1 t_2 + s_5 t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3) \\ + \frac{1}{s_3 s_5} \left(s_2 - s_4 + s_6 - \frac{s_2 s_6}{t_2} - t_2 \right) + \frac{1}{s_1 s_3} \left(-s_2 + s_4 + s_6 - \frac{s_4 s_6}{t_3} - t_3 \right) \\ + \frac{1}{s_1 s_5} \left(s_2 + s_4 - s_6 - \frac{s_2 s_4}{t_1} - t_1 \right) - \frac{1}{s_1 s_3 s_5} (s_4 t_2 + s_2 t_3 + s_6 t_1 - t_2 t_3 - t_1 t_2 - t_1 t_3) \\ - \frac{1}{s_2 s_5} \left(\frac{s_1 s_4}{t_1} + \frac{s_3 s_6}{t_2} - t_3 \right) - \frac{1}{s_3 s_6} \left(\frac{s_2 s_5}{t_2} + \frac{s_1 s_4}{t_3} - t_1 \right) - \frac{1}{s_1 s_4} \left(\frac{s_3 s_6}{t_3} + \frac{s_2 s_5}{t_1} - t_2 \right) \\ + \frac{2}{s_1} \left(1 - \frac{s_6}{t_3} - \frac{s_2}{t_1} \right) + \frac{2}{s_2} \left(1 - \frac{s_1}{t_1} - \frac{s_3}{t_2} \right) + \frac{2}{s_3} \left(1 - \frac{s_2}{t_2} - \frac{s_4}{t_3} \right) \\ + \frac{2}{s_4} \left(1 - \frac{s_3}{t_3} - \frac{s_5}{t_1} \right) + \frac{2}{s_5} \left(1 - \frac{s_4}{t_1} - \frac{s_6}{t_2} \right) + \frac{2}{s_6} \left(1 - \frac{s_5}{t_2} - \frac{s_1}{t_3} \right) - 4 \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} \right) + \dots, \quad (\text{A.14})$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(2)\alpha'^2)$ with respect to the leading (QCD) contribution. Actually, up to the last term, which is invariant under cyclic permutations, the leading contribution of (A.14) is the cyclicized version of the expansion of L_5^Y given in (A.4). Indeed L_5^Z is invariant under cyclic permutations to all orders in α' .

The functions G_i entering (3.17) are:

$$G_1^Z = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\frac{1-s_{14}}{x} - \frac{s_{14}}{1-x} \right) \frac{xy \mathcal{I}(x, y, z)}{(1-xy)(1-xyz)}, \\ G_2^Z = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\frac{1-s_{34}}{1-x} - \frac{s_{34}}{x} \right) \frac{\mathcal{I}(x, y, z)}{yz(1-x)}, \quad (\text{A.15}) \\ G_3^Z = s_{24} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{xyz(1-x)(1-xy)(1-xyz)},$$

$$\begin{aligned}
G_4^Z &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(1 - \frac{s_{12}}{x}\right) \frac{y \mathcal{I}(x, y, z)}{(1-y)(1-xy)(1-yz)(1-xyz)} , \\
G_5^Z &= s_{13} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{y \mathcal{I}(x, y, z)}{x(1-x)(1-y)(1-yz)} , \\
G_6^Z &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\frac{1-s_{23}}{x} - \frac{s_{23}}{1-x}\right) \frac{\mathcal{I}(x, y, z)}{xyz(1-y)(1-yz)} .
\end{aligned}$$

Their low-energy expansions are

$$\begin{aligned}
G_1^Z &= -\frac{s_2 + s_5 - t_1 - t_2}{s_1 s_3 t_3} + \dots , \\
G_2^Z &= \frac{1}{s_3 s_6} - \frac{1}{s_6 t_3} - \frac{2}{s_6 t_2} - \frac{s_2}{s_3 s_6 t_2} - \frac{s_3}{s_2 s_6 t_2} - \frac{s_4}{s_3 s_6 t_3} + \dots , \\
G_3^Z &= s_{24} \left(\frac{1}{s_2 s_6 t_2} + \frac{1}{s_3 s_6 t_2} + \frac{1}{s_1 s_3 t_3} + \frac{1}{s_3 s_6 t_3} \right) + \dots , \\
G_4^Z &= \frac{1}{s_1 s_4} - \frac{1}{s_4 t_3} - \frac{2}{s_4 t_1} - \frac{s_1}{s_2 s_4 t_1} - \frac{s_2}{s_1 s_4 t_1} - \frac{s_6}{s_1 s_4 t_3} + \dots , \\
G_5^Z &= s_{13} \left(\frac{1}{s_1 s_4 t_1} + \frac{1}{s_2 s_4 t_1} + \frac{1}{s_1 s_3 t_3} + \frac{1}{s_1 s_4 t_3} \right) + \dots , \\
G_6^Z &= \frac{1}{s_2 s_4} + \frac{1}{s_2 s_6} - \frac{2}{s_4 s_6} - \frac{2}{s_4 t_1} - \frac{2}{s_6 t_2} - \frac{s_1}{s_2 s_4 t_1} - \frac{s_2}{s_1 s_4 t_1} - \frac{s_2}{s_3 s_6 t_2} - \frac{s_3}{s_2 s_6 t_2} \\
&\quad - \frac{s_2}{s_1 s_3 t_3} - \frac{s_2}{s_1 s_4 t_3} - \frac{s_2}{s_3 s_6 t_3} - \frac{s_2}{s_4 s_6 t_3} - \frac{t_3}{s_2 s_4 t_6} + \dots ,
\end{aligned} \tag{A.16}$$

where dots represent terms suppressed by a factor of order $\mathcal{O}(\zeta(2)\alpha'^2)$ with respect to the leading (QCD) contribution.

Appendix B. Basis representation of the functions N_i

In Ref.[3] it was shown that the full six-gluon amplitude can be expressed in a basis of six multiple hypergeometric functions. In [4,5] we introduced a specific basis, $\{F_1, \dots, F_6\}$ to describe the MHV amplitude. These six hypergeometric functions are represented by generalized Euler integrals:

$$\begin{aligned}
F_1 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{xyz} & F_2 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{z(1-xy)} \\
F_3 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{1-xyz} & F_4 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{y \mathcal{I}(x, y, z)}{(1-xy)(1-yz)}
\end{aligned} \tag{B.1}$$

$$F_5 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{(1-xy)(1-xyz)} \quad F_6 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\mathcal{I}(x, y, z)}{(1-yz)(1-xyz)} .$$

In order to make contact between NMHV and MHV amplitudes, it would be desirable to express the NMHV functions, in particular the functions N_i that enter the final result, in terms of this basis. Indeed, some NMHV functions are related to this basis in a simple way. For example:

$$\begin{aligned} G_1^Y &= s_6 F_2 + (s_1 - s_5 - s_6 + t_2) F_3 - s_1 F_5 \quad , \quad G_2^Y = -s_3 F_1 \quad , \\ G_3^Y &= -(s_3 - t_2) F_1 + (s_6 - t_3) F_2 - (s_3 + s_6 - t_2 - t_3) F_3 - s_1 F_5 \quad , \end{aligned} \quad (\text{B.2})$$

however other relations are more complicated. Here, we focus on the functions N_i , see Eqs.(4.4), (4.8) and (4.12), which determine the NMHV amplitudes (4.3), (4.7) and (4.11), respectively.

B.1. Helicity configuration Y

$$\begin{aligned} N_1^Y &= s_1 s_3 (s_4 t_3)^{-1} \{ s_2 s_5 t_2 F_1 + s_2 s_5 (s_4 + s_5 - t_1) N_0 - [s_1 s_5 + (s_4 - t_1)(s_5 + t_3)] \tilde{N}_0 \\ &\quad - s_5 [s_3 s_4 - s_2 (s_3 - t_3)] F_2 - s_5 s_1 (s_2 - t_1) F_3 - s_1 s_4 s_5 F_5 - s_3 s_4 t_3 (F_3 - F_5) \\ &\quad + (s_5 + t_3) [s_4 t_2 F_2 + s_2 s_5 F_3 + t_1 (s_4 - t_1) F_3 + s_4 (s_4 - t_1)(F_4 - F_5) + s_4 s_5 F_4] \} \quad , \\ N_2^Y &= s_1 (s_4 t_1)^{-1} \{ s_6 t_2 F_1 + [s_6 (s_5 - t_1) - s_4 (s_5 - t_2)] N_0 - (s_1 - s_3) \tilde{N}_0 \\ &\quad + [s_5 s_6 + t_1 (s_1 - t_3)] F_3 - (s_1 - s_3)(s_4 + s_6) F_3 - s_4 s_5 (F_3 + F_4 - F_5) \\ &\quad - (s_3 - t_3) [t_2 F_3 + (s_1 - s_3) (F_3 - F_5) + (s_5 - t_1) (F_4 - F_5)] \} \quad , \\ N_3^Y &= s_3 F_1 \quad , \\ N_4^Y &= s_1 s_4^{-1} \{ t_2^2 F_1 + t_2 (s_4 + s_5 - t_1) N_0 - (s_1 - s_3 - s_5 - t_3) \tilde{N}_0 + s_3 t_2 (F_2 - F_3) \\ &\quad - t_2 t_3 F_2 + 2 s_3 s_4 F_3 + (s_1 - s_3 - s_5 - t_3) (t_1 - t_2) F_3 + s_4 (s_1 + s_3 - s_5 - t_3) (F_4 - F_5) \} \quad , \\ N_5^Y &= s_3 s_4^{-1} \{ t_1 t_2 F_1 - (s_3 s_4 - s_4 s_5 - s_4 t_3 + t_1 t_3) (F_2 - F_3) + (s_1 s_4 + s_4 s_5 + s_3 t_1) F_2 \\ &\quad + t_1 (s_4 + s_5 - t_1) N_0 + (s_1 s_4 - s_1 t_1 + s_5 t_1) F_3 \} \quad , \\ &\hspace{25em} (\text{B.3}) \\ N_6^Y &= s_4^{-1} \{ t_2 t_3 F_1 + [s_4 (s_1 + s_3 - s_5) + t_3 (s_5 - t_1)] N_0 + s_3 t_3 F_2 - t_3^2 (F_2 - F_3) \\ &\quad - (s_1 - s_5) t_3 F_3 \} \quad , \end{aligned}$$

with the definitions:

$$\begin{aligned} N_0 &= \frac{s_6}{s_2} (F_2 - F_3) \\ &\quad - \frac{s_3 - s_5 + t_1 - t_3}{s_2} (F_3 + F_4) - \frac{s_1 - s_3 + s_5 - t_1}{s_2} F_5 + \frac{s_1 + s_3 - s_5 - t_3}{s_2} F_6 \quad , \quad (\text{B.4}) \\ \tilde{N}_0 &= s_6 (F_2 - F_3) + (s_1 - s_3 + t_2) F_3 + (s_4 + s_5 - t_1) (F_4 - F_5) - (s_1 - s_3) F_5 \quad . \end{aligned}$$

B.2. Helicity configuration X

$$\begin{aligned}
N_1^X &= -s_1 s_3 \{ s_6 (F_2 + F_3) + s_1 (F_3 - F_5) + (s_5 - t_1) (F_3 + F_4 - F_5) - s_5 F_6 \}, \\
N_2^X &= s_1 (s_4 t_1)^{-1} \{ s_6 t_2 F_1 + [s_6 (s_5 - t_1) - s_4 (s_5 - t_2)] N_0 - (s_1 - s_3) \tilde{N}_0 \\
&\quad - (s_4 + s_6) (s_1 - s_3) F_3 - (s_3 - t_3)(s_5 - t_1) (F_4 - F_5) + (s_1 - s_3) (s_3 - t_3) (F_3 - F_5) \\
&\quad - s_4 s_5 (F_3 + F_4 - F_5) + [s_5 s_6 + t_1 (s_1 - t_3) - t_2 (s_3 - t_3)] F_3 \}, \\
N_3^X &= s_3 F_1, \\
N_4^X &= s_1 s_4^{-1} \{ s_4 t_2 N_0 + t_3 \tilde{N}_0 - t_1 t_3 F_3 - s_4 t_3 (F_4 - F_5) + 2 s_3 s_4 (F_3 + F_4 - F_5) \}, \\
N_5^X &= s_3 \{ t_1 N_0 + t_3 (F_2 - F_3) + 2 s_1 F_3 \}, \\
N_6^X &= s_4^{-1} \{ t_2 t_3 F_1 + [s_4 (s_1 + s_3 - s_5) + t_3 (s_5 - t_1)] N_0 + s_3 t_3 F_2 - t_3^2 (F_2 - F_3) \\
&\quad - (s_1 - s_5) t_3 F_3 \}.
\end{aligned} \tag{B.5}$$

B.3. Helicity configuration Z

$$\begin{aligned}
&s_1 s_3 s_6 (s_1 + s_2 - t_1)^{-1} (s_2 + s_3 - t_2)^{-1} N_1^Z = [s_6 (s_1 s_3 s_5 + s_2 s_4 s_6 + s_3 s_6 t_1 + s_1 s_4 t_2)] \tilde{F}_1 \\
&\quad + \{ -s_2 s_4 s_5 s_6 + [-s_3 t_1^2 + s_3 (s_1 + s_2 + s_4) t_1 + s_4 (s_2^2 + s_1 (t_2 - s_3))] s_6 + s_1 s_4 \\
&\quad \times (s_2 + s_3 - t_2) t_2 \} \tilde{F}_2 + \{ -s_2 s_6 s_5^2 - s_5 s_6 [s_2 (s_2 + s_3 + s_6 - t_1 - 2t_2) - s_1 s_3] \\
&\quad + (s_2 + s_3) s_6^2 t_1 - s_1 t_2 (s_2 + s_3 - t_2) (s_1 - s_3 + s_4 - t_1 + t_2) - s_6 [(s_4 - t_1) s_2^2 \\
&\quad - (t_1 (s_3 - 2t_2) + s_4 t_2) s_2 + s_3 t_1 (-s_3 + s_4 - t_1 + t_2) + s_1 (s_2^2 + 2(s_3 - t_2) s_2 + s_3^2 \\
&\quad + t_2^2 + s_3 (-s_4 + t_1 - 2t_2))] \} \tilde{F}_3 \\
&\quad + \{ -(s_2 + s_3 - t_2) [s_2 s_5 s_6 + (s_1 (s_2 + s_3 - t_2) - (s_2 + s_3) t_1) s_6 + s_1 t_2 (s_1 - s_3 - t_1 + t_2)] \} \tilde{F}_4 \\
&\quad + \{ s_2 s_6 s_5^2 + [-(s_2 + s_3) s_6 t_1 + (s_1 + s_2) s_6 (s_3 - t_2) + s_1 t_2 (-s_2 - s_3 + t_2)] s_5 + s_6 t_1 \\
&\quad \times ((s_1 + s_2 + s_3) t_2 - s_3 (s_2 + s_3)) + s_1 (s_2 + s_3 - t_2) (s_6 (s_2 + s_3 - t_2) + t_2 (s_1 - s_3 + t_2)) \} \tilde{F}_5 \\
&\quad + \{ (s_2 + s_3 - t_2) [(s_2 + s_3) s_6 (s_1 - t_1) + s_1 (s_1 - s_3 - t_1) t_2 + s_5 (s_2 s_6 + s_1 t_2)] \} \tilde{F}_6, \\
&s_1 s_2 s_3 t_1 (s_1 + s_2 - t_1)^{-1} N_2^Z = \{ s_2 (s_3 + s_4) s_6 t_1 + (s_1 + s_2) s_3 s_6 t_3 \} \tilde{F}_1 \\
&\quad + \{ -s_2 s_3 t_1^2 + (s_1 + s_2) s_3 (-s_1 s_2 + s_3 s_4 - s_4 s_5 + s_1 s_6 + s_2 t_3) \\
&\quad + t_1 (-s_1 s_3 s_6 + s_2 (2s_1 s_3 + 2s_3 s_4 + s_2 (s_3 + s_4) - s_4 s_5 - s_3 t_3)) \} \tilde{F}_2 \\
&\quad + \{ s_2 t_1^2 (s_2 + 2s_3 + s_5 + s_6 - 2t_2) + s_3 [-(-s_1 + s_2 + s_3 - s_4 + s_5 + s_6 - 2t_2) \\
&\quad \times (-s_3 + s_5) - (s_2 + s_5 - t_2) t_3] (s_1 + s_2) + t_1 [-s_2 s_5^2 - s_2 s_5 (s_2 + s_6 - 2t_2)], \\
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
& +s_1(-s_2^2 + s_3(s_3 + s_6 - t_2) + s_2(-2s_3 + t_2)) + s_2(2s_3^2 + s_2(s_3 - s_4) \\
& + 2s_3s_6 + s_4t_2 + s_3(-2s_4 - 3t_2 + t_3))] \} \tilde{F}_3 \\
& + \{(-s_1 + s_3 - s_5 + t_1)[s_1s_3 + s_2(s_3 + t_1)](s_2 + s_3 - t_2)\} \tilde{F}_4 \\
& + \{-s_2t_1^2(s_3 + s_5 - t_2) + (s_1 + s_2)s_3(-s_3 + s_5)(-s_1 + s_3 + s_5 - t_2) \\
& + s_2t_1[s_5^2 + s_1(s_2 + 3s_3 - t_2) + 2s_3(-s_3 + t_2) - s_5(s_3 + t_2)]\} \tilde{F}_5 \\
& + \{(s_1 - s_3 + s_5 - t_1)[s_1s_3 + s_2(s_3 + t_1)](s_2 + s_3 - t_2)\} \tilde{F}_6 ,
\end{aligned}$$

$$\begin{aligned}
s_1s_3t_3(s_2 + s_3 - t_2)^{-1} N_3^Z &= \{s_2(s_1 + s_6)t_2 + s_1(s_2 + s_3)t_3\} F_1 \\
& + \{-(s_1 + s_6)(-s_2s_3 + s_3s_4 + s_1s_6 - s_5s_6 - s_4t_2) \\
& + [-s_5s_6 + s_1(2s_3 + s_4 + 2s_6 - t_2)]t_3 - s_1t_3^2\} F_2 \\
& + \{(s_1 + s_6)[-(s_1 - s_5)(s_1 + s_2 - s_3 + s_4 + s_5 - s_6 - 2t_1) - (s_1 + s_4 - t_1)t_2] \\
& + [s_1^2 - s_5(s_4 + s_5 - s_6 - 2t_1) + s_3t_1 - s_2(s_3 + s_5 - t_2) + (s_4 + s_5 - 2t_1)t_2 \\
& + s_1(2s_2 - 2s_3 + s_4 + s_5 - 2s_6 - 3t_1 + 2t_2)]t_3 + s_1t_3^2\} F_3 \\
& + \{-(s_4 + s_5 - t_1)[(s_1 + s_3 - s_5)(s_1 + s_6) + (-2s_1 + s_5 - t_2)t_3]\} F_4 \\
& + \{(s_1 - s_5)(s_1 + s_6)(s_1 - s_3 + s_5 - t_1) - [2s_1^2 - s_2s_3 + s_3t_1 \\
& - (s_5 - t_1)(s_5 - t_2) + s_1(-3s_3 + s_5 - 2t_1 + t_2)]t_3\} F_5 \\
& + (s_4 + s_5 - t_1)[(s_1 + s_3 - s_5)(s_1 + s_6) + (-2s_1 + s_5 - t_2)t_3] F_6 ,
\end{aligned}$$

$$\begin{aligned}
s_1s_2s_3s_6(s_1 + s_2 - t_1)^{-1} N_4^Z &= \{-s_2s_3s_5s_6 - s_2s_3s_6^2 + s_1t_2[s_2(s_3 + s_4) + s_3t_3] \\
& + s_2s_6[-s_1s_3 + s_3^2 + s_2(s_3 + 2s_4) - s_4t_2 + s_3(s_4 + t_1 + t_3)]\} s_6\tilde{F}_1 \\
& + \{-s_2s_3s_6^2(s_1 + s_2 - t_1) + s_1(s_2 + s_3)s_4(s_2 + s_3 - t_2)t_2 \\
& - s_2s_5s_6[s_1s_3 + 2s_3s_4 + s_2(s_3 + 2s_4) - s_3t_1 - s_4t_2] + s_6[s_1^2s_3(-s_2 + t_2) \\
& + s_2[s_2^2(s_3 + 2s_4) + s_2(s_3^2 - s_4t_2 + s_3(3s_4 + t_3)) + s_3(s_3(2s_4 - t_1) + s_4(t_1 - t_2) \\
& - t_1(t_1 + t_3))] + s_1[s_3(s_4 - t_1)t_2 + s_2(s_3^2 + s_4t_2 + s_3(-s_4 + 2t_1 + t_2 + t_3))]\} \tilde{F}_2 \\
& + \{s_2s_6^2[s_1s_3 + s_2(3s_3 + 2t_1) + (2s_3 + t_1)(s_3 - t_2)] + s_2s_5^2s_6(-2s_2 - s_3 + t_2) \\
& - s_1(s_2 + s_3)(s_2 + s_3 - t_2)t_2(s_1 - s_3 + s_4 - t_1 + t_2) - s_2s_5s_6[2s_2^2 - 2s_1s_3 + s_3^2 \\
& + s_2(2s_3 - 2t_1 - 5t_2) + s_6(2s_2 + s_3 - t_2) + t_2(t_1 + 2t_2) + s_3(-s_4 - 3t_2 + t_3)] \\
& + s_6[s_1[s_1s_3(s_2 - t_2) - 2s_2^3 + 4s_2^2(-s_3 + t_2) + s_3t_2(s_3 - s_4 + t_1 - t_2 + t_3) \\
& - s_2[3s_3^2 + 2t_2^2 + s_3(-s_4 + 2t_1 - 4t_2 + t_3)]] + s_2[2s_3^3 + s_2^2(s_3 - 2s_4 + 2t_1) \\
& + s_3^2(-2s_4 + 3t_1 - 4t_2) - (s_4 - 2t_1)t_2^2 + s_2[3s_3^2 + (3s_4 - 5t_1)t_2 \\
& - s_3(3s_4 - 4t_1 + 4t_2 + t_3)] + s_3[t_1^2 + t_1(-s_4 - 4t_2 + t_3) + t_2(2(s_4 + t_2) + t_3)]]\} \tilde{F}_3
\end{aligned}$$

$$\begin{aligned}
& + \{(s_2 + s_3 - t_2)[s_1(s_2 + s_3)(-s_1 + s_3 + t_1 - t_2)t_2 + s_2s_5s_6(-2(s_2 + s_3) + t_2) \\
& + s_6[-2s_1s_2(s_2 + s_3) + s_2(s_3 + t_1)(2(s_2 + s_3) - t_2) + s_1(2s_2 + s_3)t_2]]\} \tilde{F}_4 \\
& + \{s_2s_5^2s_6(2(s_2 + s_3) - t_2) + s_1(s_2 + s_3)(s_2 + s_3 - t_2)t_2(s_1 - s_3 + t_2) \\
& - s_5[s_1(s_2 + s_3)(s_2 + s_3 - t_2)t_2 + s_6(s_1(s_2 + s_3)t_2 + s_2[2(s_2 + s_3) - t_2](t_1 + t_2))]] \\
& + s_6[-s_2(s_3 + t_1)(s_3 - t_2)(2(s_2 + s_3) - t_2) + s_1^2s_3t_2 + s_1[2s_2^3 + s_2^2(6s_3 - 4t_2) \\
& + s_3t_2(-s_3 + t_2) + s_2[4s_3^2 - 5s_3t_2 + t_2(t_1 + 2t_2)]]]\} \tilde{F}_5 \\
& + \{(s_1 - s_3 + s_5 - t_1)(s_2 + s_3 - t_2)[s_2s_6(2(s_2 + s_3) - t_2) + s_1(s_2 + s_3)t_2]\} \tilde{F}_6, \\
s_1s_2s_3s_6(s_2 + s_3 - t_2)^{-1} N_5^Z = & \{s_6[-s_1s_2s_4s_5 + s_2s_6(s_4(s_1 + 2s_2 - t_1) + s_3t_1) \\
& + s_1s_3t_1(s_2 + t_3) + s_1s_2s_4(s_1 + s_2 - s_3 - s_4 + t_2 + t_3)]\} \tilde{F}_1 \\
& + \{-s_2s_4s_5[2(s_1 + s_2)s_6 - s_6t_1 + s_1(s_2 + s_3 - t_2)] \\
& + s_6[(s_1 + s_2)[(s_1 + s_2)s_3 + (-s_2 + s_3)s_4]t_1 - (s_1 + s_2)s_3t_1^2 \\
& + s_2s_4(2s_1^2 + 2s_2^2 + s_1(3s_2 - s_3 + t_2))]\} \tilde{F}_2 \\
& + s_1s_4(s_2 + s_3 - t_2)[s_3t_1 + s_2(s_1 + s_2 - s_3 - s_4 + t_2 + t_3)]\} \tilde{F}_2 \\
& + \{s_2s_6^2(s_1 + 2s_2 + s_3 - t_1)t_1 + s_2s_5^2s_6(-s_1 - 2s_2 + t_1) \\
& - s_1(s_2 + s_3 - t_2)(s_1 - s_3 + s_4 - t_1 + t_2)[s_3t_1 + s_2(s_1 + s_2 - s_3 - s_4 + t_2 + t_3)] \\
& - s_2s_5[s_6^2(s_1 + 2s_2 - t_1) - s_1(s_2 + s_3 - t_2)(s_1 - s_3 + s_4 - t_1 + t_2) \\
& + s_6[-s_1^2 + (2s_2 - t_1)(s_2 + s_3 - t_1 - 2t_2) + s_1(s_2 + s_3 - s_4 - 3t_2 + t_3)]] \\
& + s_6[s_1^2(-(s_2 + s_3)t_1 + s_2(s_2 + s_3 - t_2)) - s_2[2s_2^2(s_4 - t_1) + t_1[-s_2^2 + (s_4 - 2t_1)t_2 \\
& + s_3(s_4 + t_2)]] + s_2(-s_4(t_1 + 2t_2) + t_1(-2s_3 + t_1 + 4t_2))] + s_1[-s_2^2(s_3 + s_4 - t_2) \\
& + s_3t_1(s_3 - s_4 + t_1 - t_2 + t_3) + s_2[-s_3^2 - 2t_1t_2 - t_2^2 + s_4(-t_1 + t_2) + s_3(s_4 + 2t_2) \\
& + t_1(t_1 + t_3)]]]\} \tilde{F}_3 \\
& + \{(s_2 + s_3 - t_2)[s_2s_5[s_6[-2(s_1 + s_2) + t_1] + s_1(s_1 - s_3 - t_1 + t_2)] \\
& + s_6[s_1^2s_2 + s_2(2s_2 + s_3 - t_1)t_1 + s_1(s_3t_1 + s_2(-s_3 + t_1 + t_2))]] \\
& - s_1(s_1 - s_3 - t_1 + t_2)(s_3t_1 + s_2(s_1 + s_2 - s_3 - s_4 + t_2 + t_3))]\} \tilde{F}_4 \\
& + \{(-s_1 + s_3 + s_5 - t_2)[s_2s_5[2(s_1 + s_2)s_6 - s_6t_1 + s_1(s_2 + s_3 - t_2)] \\
& + s_6[-(s_1 + s_2)(2s_2 + s_3)t_1 + s_2t_1^2 + s_1s_2(s_2 + s_3 - t_2)] \\
& - s_1(s_2 + s_3 - t_2)(s_3t_1 + s_2(s_1 + s_2 - s_3 - s_4 + t_2 + t_3))]\} \tilde{F}_5 \\
& + \{-(s_2 + s_3 - t_2)[s_1s_2s_5^2 - s_2s_6(s_1 - t_1)(s_1 + 2s_2 + s_3 - t_1) \\
& - s_5[s_2s_6(s_1 + 2s_2 - t_1) + s_1(s_2 + s_3)t_1 + s_1s_2(s_2 - s_4 + t_2 + t_3)] \\
& - s_1(s_1 - s_3 - t_1)(s_3t_1 + s_2(s_1 + s_2 - s_3 - s_4 + t_2 + t_3))]\} \tilde{F}_6
\end{aligned}$$

,

$$\begin{aligned}
s_1 s_2 s_3 s_6 N_6^Z = & s_6 \left\{ -s_2 s_3 s_6^2 + s_1 (s_2 - s_4 + t_3) (s_2 (s_3 + s_4) + s_3 t_3) + s_6 [-s_1 s_3 (s_2 + t_3) \right. \\
& + s_2 (s_2 (s_3 + 2s_4) + s_4 s_5 - s_4 (t_1 + t_2) + s_3 t_3)] \left. \right\} \tilde{F}_1 \\
& + \left\{ -(s_1 + s_2) s_3 s_6^2 (s_1 + s_2 - t_1) + s_1 (s_2 + s_3) s_4 (s_2 + s_3 - t_2) (s_2 - s_4 + t_3) \right. \\
& + s_6 [s_1^2 (s_2 (s_3 + s_4) + s_3 t_3) + s_2 [s_2^2 (s_3 + 2s_4) - s_4 (-s_3 + s_5) (s_3 + s_5 - t_2) \\
& - s_2 (s_4 s_5 + s_4 (t_1 + t_2) + s_3 (-2s_4 + t_1 - t_3)) + t_1 (s_4 s_5 - s_3 t_3)] + s_1 [2s_2^2 (s_3 + s_4) \\
& - s_2 (s_3 + s_4) t_1 + 2s_2 s_3 t_3 + s_3 (s_4 s_5 - t_1 t_3)] \left. \right\} \tilde{F}_2 \\
& + \left\{ -s_2 s_5^3 s_6 + s_5^2 s_6 [s_1 s_3 - s_2 (3s_2 + s_3 + s_6 - 2t_1 - 3t_2)] \right. \\
& + s_6^2 [-s_1 s_3 (-s_2 + s_3 + t_1) + s_2 (s_3^2 + s_2 (3s_3 + 2t_1) - t_1 (t_1 + t_2) - s_3 (t_1 + 2t_2))] \\
& - s_1 (s_2 + s_3) (s_2 + s_3 - t_2) (s_1 - s_3 + s_4 - t_1 + t_2) (s_2 - s_4 + t_3) \\
& + s_5 s_6 [s_1^2 s_2 + s_1 (s_3 s_6 + s_3 (s_3 - t_1 - 2t_2) + s_2 (2s_3 + s_4 - t_1 + t_2 - t_3)) \\
& + s_2 [-2s_2^2 + s_3^2 - s_2 (s_3 + s_4 - 4t_1 - 5t_2) + (s_4 - 2t_2) t_2 + s_6 (-2s_2 + s_3 + 2t_1 + t_2) \\
& - t_1 (t_1 + 5t_2) - s_3 (-2t_1 + t_3)] - s_6 [s_1^3 s_3 + s_1^2 [-s_2^2 + s_2 (t_1 + t_2) + s_3 (s_4 - t_1 + t_2 - t_3)] \\
& + s_1 [s_3^3 + s_2 (s_2 (t_1 - t_2) + t_2^2 + t_1 (s_4 - t_1 - t_3)) + s_3^2 (s_4 + t_1 - t_2 - 2t_3) \\
& + s_3 (s_2 (s_4 + t_1 + t_2 - 2t_3) + t_2 (-2t_1 + t_3))] + s_2 (-s_3^3 - s_2^2 (s_3 - 2s_4 + 2t_1) \\
& + (s_4 - 2t_1) t_2 (t_1 + t_2) + s_3^2 (s_4 + 3t_2) + s_3 [t_1^2 + t_1 (t_2 - t_3) - t_2 (s_4 + 2t_2 + t_3)] \\
& + s_2 (-3s_3^2 - s_4 (t_1 + 3t_2) + t_1 (t_1 + 5t_2) + s_3 (2s_4 - 3t_1 + 4t_2 + t_3))] \left. \right\} \tilde{F}_3 \\
& + \left\{ (s_2 + s_3 - t_2) [-s_2 s_5^2 s_6 + s_5 s_6 (s_1 (-s_2 + s_3) + s_2 (-2s_2 + 2t_1 + t_2)) \right. \\
& + s_6 [s_1^2 (s_2 + 2s_3) + s_2 (s_3 + t_1) (2s_2 + s_3 - t_1 - t_2) + s_1 (-2s_3 t_1 + s_2 (2s_3 + t_2))] \\
& + s_1 (s_2 + s_3) (-s_1 + s_3 + t_1 - t_2) (s_2 - s_4 + t_3)] \left. \right\} \tilde{F}_4 \\
& + \left\{ (-s_1 + s_3 + s_5 - t_2) [s_2 s_5^2 s_6 + s_5 s_6 (s_1 (s_2 - s_3) + s_2 (2s_2 - 2t_1 - t_2)) \right. \\
& - s_6 (s_1^2 s_3 + s_2 (s_3 + t_1) (2s_2 + s_3 - t_1 - t_2) + s_1 (-s_2^2 - s_3 t_1 + s_2 (s_3 + t_1 + t_2))] \\
& - s_1 (s_2 + s_3) (s_2 + s_3 - t_2) (s_2 - s_4 + t_3)] \left. \right\} \tilde{F}_5 \\
& + \left\{ (s_1 - s_3 + s_5 - t_1) (s_2 + s_3 - t_2) [s_2 s_5 s_6 + s_6 [-s_1 s_3 + s_2 (2s_2 + s_3 - t_1 - t_2)] \right. \\
& + s_1 (s_2 + s_3) (s_2 - s_4 + t_3)] \left. \right\} \tilde{F}_6 ,
\end{aligned}$$

with:

$$\begin{aligned}
\tilde{F}_3 &= F_3 + F_4 - F_6 , \quad \tilde{F}_4 = F_6 , \quad \tilde{F}_5 = F_4 , \quad \tilde{F}_6 = F_5 , \\
\tilde{F}_2 &= \frac{s_6}{s_4} F_2 + \left(1 + \frac{s_1}{s_4} - \frac{s_6}{s_4} - \frac{t_1}{s_4} + \frac{t_2}{s_4} - \frac{s_3}{s_4} \right) F_3 + \left(1 + \frac{s_5}{s_4} - \frac{t_1}{s_4} \right) F_4 \\
&+ \left(-\frac{s_1}{s_4} - \frac{s_5}{s_4} + \frac{t_1}{s_4} + \frac{s_3}{s_4} \right) F_5 - F_6 ,
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\tilde{F}_1 = & \frac{s_2 t_2}{s_4 t_3} F_1 + \left(-\frac{s_2}{s_4} + \frac{s_3 s_2}{s_4 t_3} - \frac{s_3}{t_3} + \frac{s_5 s_6}{s_4 t_3} - \frac{s_1 s_6}{s_4 t_3} + \frac{t_2}{t_3} \right) F_2 \\
& + \left(-\frac{s_1^2}{s_4 t_3} + \frac{s_6 s_1}{s_4 t_3} + \frac{2 t_1 s_1}{s_4 t_3} - \frac{t_2 s_1}{s_4 t_3} - \frac{s_2 s_1}{s_4 t_3} + \frac{s_3 s_1}{s_4 t_3} - \frac{s_1}{t_3} + \frac{s_5}{s_4} - \frac{t_1}{s_4} + \frac{s_2}{s_4} + \frac{s_5^2}{s_4 t_3} \right. \\
& + \left. \frac{s_2 s_5}{s_4 t_3} - \frac{s_3 s_5}{s_4 t_3} + \frac{s_5}{t_3} - \frac{s_5 s_6}{s_4 t_3} - \frac{2 s_5 t_1}{s_4 t_3} + \frac{t_1 t_2}{s_4 t_3} - \frac{t_2}{t_3} \right) F_3 \\
& + \left(\frac{s_5^2}{s_4 t_3} + \frac{s_5}{s_4} - \frac{t_1 s_5}{s_4 t_3} - \frac{s_1 s_5}{s_4 t_3} - \frac{s_3 s_5}{s_4 t_3} + \frac{s_5}{t_3} - \frac{t_1}{s_4} - \frac{s_1}{t_3} - \frac{s_3}{t_3} + \frac{s_1 t_1}{s_4 t_3} + \frac{s_3 t_1}{s_4 t_3} \right) F_4 \\
& + \left(\frac{s_1^2}{s_4 t_3} - \frac{t_1 s_1}{s_4 t_3} - \frac{s_3 s_1}{s_4 t_3} - \frac{s_5^2}{s_4 t_3} + \frac{s_3 s_5}{s_4 t_3} + \frac{s_5 t_1}{s_4 t_3} \right) F_5 \\
& + \left(-\frac{s_5^2}{s_4 t_3} - \frac{s_5}{s_4} + \frac{t_1 s_5}{s_4 t_3} + \frac{s_1 s_5}{s_4 t_3} + \frac{s_3 s_5}{s_4 t_3} - \frac{s_5}{t_3} + \frac{t_1}{s_4} + \frac{s_1}{t_3} + \frac{s_3}{t_3} - \frac{s_1 t_1}{s_4 t_3} - \frac{s_3 t_1}{s_4 t_3} \right) F_6 .
\end{aligned}$$

Finally, we list the next order of the expansions (4.13). It comes with a factor of $\zeta(2)$:

$$\begin{aligned}
N_1^Z = & \dots + \zeta(2) (s_1 + s_2 - t_1)(s_2 + s_3 - t_2) \left\{ -\frac{t_1^2}{s_1 s_4} + \frac{t_1}{s_1} + \frac{t_1}{s_4} - \frac{s_3 t_1}{s_1 t_3} - \frac{s_6 t_1}{s_4 t_3} - \frac{t_2^2}{s_3 s_6} \right. \\
& \left. - \frac{s_2 s_5}{s_1 s_3} + \frac{t_2}{s_3} + \frac{t_2}{s_6} - \frac{s_2 s_5}{s_4 s_6} - \frac{s_2 s_4}{s_1 t_3} - \frac{s_1 s_5}{s_4 t_3} - \frac{s_2 s_6}{s_3 t_3} - \frac{s_1 t_2}{s_3 t_3} - \frac{s_4 t_2}{s_6 t_3} - \frac{s_3 s_5}{s_6 t_3} - 1 \right\} + \dots , \\
N_2^Z = & \dots + \zeta(2) (s_1 + s_2 - t_1) \left\{ -\frac{s_1}{s_4 t_1} - \frac{s_6}{s_2 s_4} + \frac{2}{s_4} - \frac{s_5}{s_2 t_1} - \frac{s_2}{s_4 t_1} - \frac{s_6}{s_3 t_3} - \frac{s_6}{s_4 t_3} - \frac{s_5}{s_3 s_1} \right. \\
& \left. - \frac{t_1}{s_4 s_1} - \frac{t_3}{s_4 s_1} - \frac{s_5}{t_1 s_1} - \frac{s_3}{t_3 s_1} - \frac{s_4}{t_3 s_1} + \frac{2}{s_1} \right\} + \dots , \\
N_3^Z = & \dots + \zeta(2) (s_2 + s_3 - t_2) \left\{ -\frac{s_1}{s_3 t_3} - \frac{t_2}{s_3 s_6} - \frac{t_3}{s_3 s_6} + \frac{2}{s_3} - \frac{s_4}{s_2 s_6} + \frac{2}{s_6} - \frac{s_5}{s_2 t_2} - \frac{s_5}{s_3 t_2} \right. \\
& \left. - \frac{s_2}{s_6 t_2} - \frac{s_3}{s_6 t_2} - \frac{s_6}{s_3 t_3} - \frac{s_4}{s_6 t_3} - \frac{s_5}{s_3 s_1} - \frac{s_4}{t_3 s_1} \right\} + \dots , \\
N_4^Z = & \dots + \zeta(2) (s_1 + s_2 - t_1) \left\{ -\frac{s_3^2}{s_1 t_3} - \frac{t_1 s_3}{s_1 s_4} + \frac{s_3}{s_1} + \frac{s_3}{s_4} - \frac{s_2 s_3}{s_1 t_3} - \frac{s_4 s_3}{s_1 t_3} + \frac{s_5 s_3}{s_1 t_3} + \frac{s_6 s_3}{s_1 t_3} - \frac{s_6 s_3}{s_4 t_3} \right. \\
& - \frac{t_1 s_3}{s_1 t_3} - \frac{t_2 s_3}{s_6 t_3} + \frac{s_3}{t_3} - \frac{t_1^2}{s_1 s_4} - \frac{t_2^2}{s_2 s_6} + \frac{2 s_2}{s_1} - \frac{2 s_5}{s_1} - \frac{s_5}{s_4} - \frac{2 s_6}{s_4} + \frac{s_5 t_1}{s_1 s_4} + \frac{s_6 t_1}{s_1 s_4} + \frac{t_1}{s_1} - \frac{s_2 t_1}{s_1 s_4} \\
& + \frac{2 t_1}{s_4} - \frac{t_1 t_2}{s_2 s_4} - \frac{t_2}{s_1} + \frac{t_2}{s_2} + \frac{t_2}{s_4} - \frac{s_2 t_2}{s_4 s_6} + \frac{2 t_2}{s_6} - \frac{t_1 t_3}{s_1 s_4} - \frac{t_2 t_3}{s_4 s_6} + \frac{t_3}{s_4} - \frac{s_1}{s_4} + \frac{s_2}{s_4} + \frac{s_6^2}{s_4 t_3} \\
& - \frac{2 s_2 s_4}{s_1 t_3} + \frac{s_5 s_6}{s_4 t_3} + \frac{s_1 s_6}{s_4 t_3} - \frac{s_2 s_6}{s_4 t_3} - \frac{s_6}{t_3} - \frac{s_6 t_1}{s_4 t_3} + \frac{s_4 t_2}{s_1 t_3} - \frac{s_1 t_2}{s_4 t_3} - \frac{s_4 t_2}{s_6 t_3} - 1 - \frac{t_2^2}{s_6 s_3} - \frac{2 s_2 s_5}{s_1 s_3} \\
& \left. + \frac{s_5 t_2}{s_1 s_3} + \frac{t_2}{s_3} - \frac{2 s_2 s_6}{t_3 s_3} - \frac{s_1 t_2}{t_3 s_3} + \frac{s_6 t_2}{t_3 s_3} \right\} + \dots ,
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
N_5^Z = & \dots + \zeta(2) (s_2 + s_3 - t_2) \left\{ -\frac{s_1^2}{s_3 t_3} - \frac{t_2 s_1}{s_3 s_6} + \frac{s_1}{s_3} + \frac{s_1}{s_6} + \frac{s_4 s_1}{s_3 t_3} + \frac{s_5 s_1}{s_3 t_3} - \frac{s_6 s_1}{s_3 t_3} - \frac{t_1 s_1}{s_4 t_3} \right. \\
& - \frac{t_2 s_1}{s_3 t_3} - \frac{s_2 s_1}{s_3 t_3} - \frac{s_4 s_1}{s_6 t_3} + \frac{s_1}{t_3} - \frac{t_1^2}{s_2 s_4} - \frac{t_2^2}{s_3 s_6} - \frac{2s_5}{s_3} + \frac{t_1}{s_2} - \frac{t_1}{s_3} + \frac{2t_1}{s_4} - \frac{s_2 t_1}{s_4 s_6} + \frac{t_1}{s_6} - \frac{t_1 t_2}{s_2 s_6} \\
& + \frac{t_2}{s_3} + \frac{s_4 t_2}{s_3 s_6} + \frac{s_5 t_2}{s_3 s_6} - \frac{s_2 t_2}{s_3 s_6} + \frac{2t_2}{s_6} - \frac{t_1 t_3}{s_4 s_6} - \frac{t_2 t_3}{s_3 s_6} + \frac{t_3}{s_6} + \frac{2s_2}{s_3} + \frac{s_2}{s_6} - \frac{s_3}{s_6} - \frac{2s_4}{s_6} - \frac{s_5}{s_6} - \frac{s_4}{t_3} \\
& - \frac{2s_2 s_6}{s_3 t_3} + \frac{s_6 t_1}{s_3 t_3} - \frac{s_6 t_1}{s_4 t_3} - \frac{s_3 t_1}{s_6 t_3} - \frac{s_4 t_2}{s_6 t_3} + \frac{s_4^2}{s_6 t_3} - \frac{s_2 s_4}{s_6 t_3} + \frac{s_3 s_4}{s_6 t_3} + \frac{s_4 s_5}{s_6 t_3} - 1 - \frac{t_1^2}{s_4 s_1} - \frac{2s_2 s_5}{s_3 s_1} \\
& \left. + \frac{s_5 t_1}{s_3 s_1} + \frac{t_1}{s_1} - \frac{2s_2 s_4}{t_3 s_1} - \frac{s_3 t_1}{t_3 s_1} + \frac{s_4 t_1}{t_3 s_1} \right\} + \dots, \\
N_6^Z = & \dots + \zeta(2) \left\{ -\frac{s_2^2}{s_4 s_6} - \frac{2s_5 s_2}{s_1 s_3} - \frac{t_1 s_2}{s_1 s_4} - \frac{t_2 s_2}{s_3 s_6} - \frac{2t_3 s_2}{s_4 s_6} + \frac{2s_2}{s_1} + \frac{2s_2}{s_3} + \frac{3s_2}{s_4} + \frac{3s_2}{s_6} - \frac{s_3 s_2}{s_1 t_3} \right. \\
& - \frac{2s_4 s_2}{s_1 t_3} - \frac{2s_6 s_2}{s_3 t_3} - \frac{s_6 s_2}{s_4 t_3} - \frac{s_1 s_2}{s_3 t_3} - \frac{s_1 s_2}{s_4 t_3} - \frac{s_3 s_2}{s_6 t_3} - \frac{s_4 s_2}{s_6 t_3} - \frac{s_5^2}{s_1 s_3} - \frac{t_3^2}{s_4 s_6} - \frac{3s_6}{s_4} + \frac{s_5 t_1}{s_1 s_3} \\
& + \frac{s_6 t_1}{s_1 s_4} - \frac{t_1}{s_3} - \frac{t_1}{s_4} + \frac{s_5 t_2}{s_1 s_3} - \frac{t_2}{s_1} + \frac{s_4 t_2}{s_3 s_6} - \frac{t_2}{s_6} - \frac{t_1 t_3}{s_1 s_4} - \frac{t_2 t_3}{s_3 s_6} + \frac{3t_3}{s_4} + \frac{3t_3}{s_6} - \frac{s_1}{s_4} - \frac{s_3}{s_6} \\
& - \frac{3s_4}{s_6} + \frac{s_6^2}{s_4 t_3} + \frac{s_1}{t_3} + \frac{s_3}{t_3} + \frac{s_1 s_4}{s_3 t_3} - \frac{s_4 s_5}{s_1 t_3} + \frac{s_3 s_6}{s_1 t_3} - \frac{s_5 s_6}{s_3 t_3} + \frac{s_1 s_6}{s_4 t_3} + \frac{s_4 t_1}{s_1 t_3} + \frac{s_6 t_1}{s_3 t_3} + \frac{s_4 t_2}{s_1 t_3} \\
& \left. + \frac{s_6 t_2}{s_3 t_3} + \frac{s_4^2}{s_6 t_3} + \frac{s_3 s_4}{s_6 t_3} + \frac{s_1}{s_2} + \frac{s_3}{s_2} + \frac{s_5}{s_2} + \frac{s_6 t_1}{s_4 s_2} + \frac{s_4 t_2}{s_6 s_2} - \frac{t_1 t_3}{s_4 s_2} - \frac{t_2 t_3}{s_6 s_2} \right\} + \dots.
\end{aligned}$$

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